

CUT LOCI IN BORDERED AND UNBORDERED

RIEMANNIAN MANIFOLDS

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Preface

If one moves starting from some point p on a normalized geodesic $c(t)$ in a Riemannian manifold, then it is a very natural question: "How long is this geodesic a minimal¹⁾ join to the starting point?" If there exists a finite number $s := \sup \{t/d(p,c(t))=t\}$ then $c(s)$ is called a cut point of p with respect to $c(t)$, $d(,)$ being the distance in the Riemannian manifold. The union of all cut points of p with respect to all geodesics starting in p is called cut locus C_p of p . The concept of cut locus with a very similar notation "ligne de partage" goes back to H. Poincaré [57]. However, prior to [57] the concept of cut locus occurs at least implicitly in a paper of H. von Mangoldt [47].

Since the fundamental articles of S.B. Myers [55] and J.H.C. Whitehead [74] it is well known that the cut locus of a point can be viewed as a geometric natural glueing seam where a cell is glued to the manifold. Or vice versa the manifold is got by attaching a cell to the cut locus. Hence the cut locus of some point contains the topological complexity of a Riemannian or more general of a Finsler manifold. Therefore since those basic papers of Myers and Whitehead the cut locus has been an important tool and also an object of many investigations in Riemannian geometry. We mention here only few names H. Rauch [58], M. Berger [14], W. Klingenberg [40], [41]. A good survey for the literature

1) Minimal join means a shortest path joining the points.

until 1966 is provided by the article of Kobayashi [44], see also chapter 5 in the book of J. Cheeger and D. Ebin [25]. Further we mention the work around the Blaschke-conjecture, see e.g. [15] and [32]. See e.g. T. Sakai [60] and M. Takeuchi [67] for investigations on cut loci in symmetric spaces. The recent survey article of T. Sakai [61] contains many references for articles related to cut loci. For questions on the triangulability of cut loci and differential topological aspects, see e.g. [31], [70], [19]. There exist also articles studying cut loci of submanifolds see e.g. R. Thom [68]. Finally we wish to mention the recent work of E. Kaufmann on cut loci of knots [37]. There exist applications for cut loci (under the name symmetric axis) in the theory of pattern recognition, see H. Blum [18]. Via the concept of Maxwell sets there exist relations between cut loci and physics, see R. Thom [69], [68].

In this paper we treat a different topic. In the center of our considerations are bordered Riemannian manifolds. We view bordered Riemannian manifolds as metric spaces with an interior metric in the sense of W. Rinow, see § 2. The distance between any two points is defined as infimum of the lengths for all paths joining those points, where the paths must be contained in the bordered manifold. The boundary has here the effect of an obstacle. Minimal joins may bifurcate at boundary points. Therefore even very elementary problems concerning regularity and local

uniqueness of locally shortest paths are here not so easy to solve. In general we do not have here a global or only a local exponential map. We meet new problems and new phenomena, see § 3, § 1, § 6. Nonetheless it is yet possible to prove some results for cut loci in bordered manifolds and just this is the main intention of our paper. A detailed summary of our results is given in the introduction in § 1.

In § 3 we discuss several definitions for the cut locus of a closed set in a bordered manifold M . We say that a point $p \in M \setminus (\partial M \cup A)$ is contained in the cut locus C_A of a closed set A if there exists a minimal join from A to p which cannot be extended as a minimal join beyond p . The cut locus C_A of A is then defined as the closure of all such points. We prove in § 3 under weak regularity assumptions for the Riemannian metric ¹⁾ as well as for the boundary ∂M , that the complement of C_A in $M \setminus (\partial M \cup A)$ is the maximal open set in $M \setminus (\partial M \cup A)$ where the distance function $d(A, \cdot)$ is C^1 -smooth. This yields as corollaries characterisations of the cut locus in unbordered manifolds. By the absence of an exponential map we are forced to find a way to "simulate conjugate points" in the cut locus. This leads us also to the concept of what we call "Lipschitz point". For this, we study points $q \in M \setminus (\partial M \cup A)$ which have the property that there exists a minimal join from q to A which can be extended as a minimal join beyond q . Such a point is called

1) We assume the Riemannian metric to be Lipschitz continuous.

extender relative to A . We prove in § 4 that a point q being an extender relative to A can be characterized as follows: "The initial vector of a minimal join from q to A differs from the initial vectors of those minimal joins going from the points in a neighbourhood of q to A by a difference which is controlled by a Lipschitz condition." In § 5 we apply the results of § 3 and § 4 to prove some partly well known results. We give a new proof of Jacobi's theorem, which says: "A geodesic is not any longer a minimal join after the first conjugate point." We show in § 5 that our concept of cut loci is a natural frame for results of Federer, Bangert, Kleinjohann and R. Walter concerning EFP-sets. As an illustration of the results and methods in §§ 2, 3, 4 we study in § 6 cut loci in bordered surfaces. Here we make only weak assumptions for the boundary. Namely we assume that the boundary consists of locally rectifiable curves. We investigate closed bordered subsurfaces of an unbordered, simply connected, complete two-dimensional Riemannian manifold M , M without conjugate points ¹⁾. We prove that a subsurface S of M is simply connected iff there exists a point $p \in S \setminus \partial S$ such that for the cut locus C_p of p is $C_p \cap (S \setminus \partial S) = \emptyset$ or iff for all $p \in S$ is $C_p = \emptyset$. This holds iff any two points in S

1) In other words M is diffeomorphic to \mathbb{R}^2 and M is complete and has no conjugate points.

can be joined by a unique normalized minimal join, or iff for all $p \in S$ the distance function $d(p, \cdot)$ is C^1 -smooth on $S \setminus (\partial S \cup \{p\})$ or iff there exists a point $p \in S \setminus \partial S$ such that $d(p, \cdot)$ is C^1 -smooth on $S \setminus (\partial S \cup \{p\})$. In case the ambient space M has nowhere positive curvature and if the subsurface S is isotopic to a circle, then the cut locus C_p of every point $p \in S$ is homeomorphic to one of the intervals $[0,1]$, $[0,1[$ and $C_p \setminus \partial S$ is C^1 -diffeomorphic to $]0,1[$. Here C_p is homeomorphic to $[0,1]$ if S is compact.

We hope that it will not cause any confusion that we do not use the standard methods from Riemannian geometry like Jacobi fields and comparison theorems. We use mainly elementary local analysis on an open set in the chart space corresponding to a neighbourhood in the Riemannian manifold. There we make estimations of the Riemannian distance in terms of the Euclidean metric induced by the local coordinates. We also identify tangent vectors with vectors. Therefore we estimate the difference of those tangent vectors in the Euclidean norm induced by the local coordinates.

In 1978 we started to work on questions related to the topic of this paper, see [75] p. 52. In those days as far as we know there was only a group of mathematicians at the University of Illinois (Urbana) working on similar questions. This group in Urbana includes S. and R. Alexander, J.D. Berg and R.L. Bishop, see [2], [3], [4], [5], [12], [13]. Meanwhile also from other people there have come many new important contributions to this field. In particular there have come contributions from V.I. Arnold in Moscow, [6], [7], [8], [9], [10], and there have come contributions from a group of analysts in Pisa including E. De Giorgi, A. Marino, D. Scolozzi and M. Tosques [48], [24], [48], [49], [62], [63]. In our introduction we shall give more details on those articles. Among the contributions mentioned above the work of the group in Urbana has probably the closest relations to the problems treated in this paper.

The author takes pleasure in expressing his hearty thanks to D. Ferus for supporting this work and for numerous helpful suggestions. In particular his suggestions have led to much clearer presentations of Lemma 4.1 and it's proof and of many other results in § 4. D. Ferus has also pointed out a serious gap in a previous version of theorem 4.5. At this place

the author also wishes to thank the above mentioned mathematicians in Illinois and Pisa for invitations to Urbana (1979) and to Pisa (1983). We also wish to thank K. Leichtweiss for an invitation to Stuttgart (1982) and D. Koutroufiotis and E. Kaufmann for several encouraging conversations.

§ 1 INTRODUCTION

Summary of results and related work of other authors

Manifolds with boundary have been subject of intense research by topologists during the last decades. Not much has been written down concerning Riemannian structures and the associated distance geometry in bordered manifolds. We think that the concept of interior metric as developed by Busemann, Rinow et al. is a canonical frame for the investigations of bordered Riemannian manifolds. Precisely we mean the following:

Let M be an n -dimensional bordered submanifold of an n -dimensional unbordered complete Riemannian manifold \hat{M} . We assume that the boundary ∂M of M is an $(n-1)$ -dimensional topological submanifold in \hat{M} . Defining the distance between two points p, q in M by

$d(p, q) := \inf\{\text{length } c \mid c \text{ a rectifiable path in } M \text{ from } p \text{ to } q\}$ we require (M, d) to be a complete, locally compact metric space. Now (M, d) is a space with an interior metric in the sense of Rinow, see § 2, remark 2.1.

Further since (M, d) is assumed to be locally compact and complete any two points in (M, d) can be joined by a distance realizing path, see [77] p. 8¹⁾. Let us call such a space (M, d) a space of type (Γ) . Note we view here a boundary component as an obstacle. The shortest path forced to stay within the bordered manifold may bend around

1) See also [59], p. 141.

the obstacle. Hence the geometry of the shortest path is greatly effected by the inflections at the obstacle. Therefore the topic described here is in the literature also called: Riemannian Obstacle Problem, see [5], Problems of By-Passing Obstacles [10], p. 63, Geodetiche con ostacolo [48]. Investigating shortest paths and the intrinsic distance function we try to show that the classical concept of cut loci carries over in a reasonable way to Riemannian manifolds with boundary and can serve as a useful instrument to study relations between geometrical and topological properties of certain bordered manifolds. Thinking of the adequate transfer of those elementary concepts like cut and conjugate points into a situation where the classical proof technics do not work any longer gave us two by-products. First we get within our setting old and new results valid and perhaps interesting also for unbordered manifolds see §§ 3, 4, 5. Second and perhaps equally important we are led to new insights of those concepts we tried to transfer. We mention here as an example a special case of theorem 5.2: "A geodesic starting in a point p is no longer minimal after a conjugate point q ." Proofs of this classical theorem due to Jacobi seem always to rely essentially on variation technics along the whole geodesic from p to q . In our setting this result is proved without variation technics using merely local considerations in any sufficiently small neighbourhood around q . It turns out that for this result and for results on the regularity of the distance function in § 3

and for all results in §§ 4, 6 and most results in § 5 it is only important how the geodesics arrive locally at the end points and it is irrelevant that those geodesics say during all their way are solutions of a differential equation. This observation is crucial for our work here. Namely in a bordered manifold the locally shortest paths henceforth also called geodesics show a branching behaviour when they come into contact with say concave boundary. Therefore in our situation we are forced to drop the assumption that the geodesics during all their way are solutions of the geodesics differential equation related to the Riemannian metric in \hat{M} or even more that the geodesics can be described by an exponential map. In contrast recall that the usual proofs for the regularity of the distance function exploit say roughly the differentiable dependence between solutions and initial values of the related differential equation. Therefore there one needs the minimal geodesics to be solutions of the geodesic differential equation during all their way. An application illuminating the just mentioned insights is the following result, a special case of theorem 6.2: "Let (M, d) be a simply connected, closed surface of the Euclidean plane, ∂M consisting of locally rectifiable paths. Then for any $p \in M$ the distance function $d(p, \cdot)$ is C^1 -smooth on $M \setminus (\partial M \cup \{p\})$ and has a locally Lipschitz continuous gradient there."

Next we try to describe some insights contributing to understand the background of the following result proved by us in [76] Lemma 2: "For any complete unbordered Riemannian manifold M and any point $p \in M$, the points with at least two minimal joins to p are dense in the cut locus C_p of p ." Our proof of Lemma 2 in [76] using the invariance of domain theorem together with a geometric construction relied on the continuity of the map

$$S : \{x \mid x \in T_p M, |x|=1\} \rightarrow [0, \infty] .$$

$$S(x) := \sup \{ \alpha \in \mathbb{R} \mid d(p, \exp_p(\alpha x)) = \alpha \}$$

$[0, \infty]$ the one point compactification of the interval $[0, \infty[$. That proof was topologically motivated and we always preferred a topological interpretation of lemma 2 in [76]. For a compact manifold this interpretation can be condensed in the following lemma 3a.1' in § 3a:

"Let M be a compact bordered or unbordered topological manifold, $f: D \xrightarrow{1)} \rightarrow M$ a continuous surjective map, f injective on $D \setminus \partial D$. Then the proper identification points of the map f i.e. the points $q \in f(D)$ with 2) $\text{card } f^{-1}(q) \geq 2$ are dense in the glueing seam $f(\partial D)$ ". We view here $f(\partial D)$ as the glueing seam where the disc D via identification on its boundary is glued to become the manifold M .

In our interpretation of lemma 2 in [76] the exponential

1) $D := \{x \in \mathbb{R}^n \mid |x| \leq 1\}$

2) $\text{card } f^{-1}(q)$ being the number of points in $f^{-1}(q)$.

map \exp_p is closely related to the above map f in lemma 3a.1', and the cut locus of p corresponds to $f(3D)$.

For compact M Lemma 2 in [76]

is an immediate consequence of the mere topological lemma 3a.1'. We believe that from a view point of global differential geometry the topological cut locus interpretation in a sense close to lemma 3a.1' is possible in a natural way also in bordered manifolds. Now we describe our intuitive geometric understanding of the cut locus in the following partly physically motivated heuristic concept which we use to prove results and motivate conjectures:

"In a bordered manifold where in general no globally defined exponential map exists we can take (as well as in the unbordered case) as cut locus the points where the distance wave fronts ¹⁾ (relative to some source set) have self interference. Even here it seems possible ²⁾ to interpret the cut locus of a point ³⁾ as a geometric natural glueing seam where a homotopically simple (perhaps even a cell-like) space is glued to become the manifold."

1) These are the level surfaces of the distance function.

2) At least in a large class of examples including bordered surfaces.

3) If A is an appropriate subset of M the analogue of this heuristic roughly says that A is in M a deformation retract of $M \setminus C_A$.

We discuss in § 3 several partly different definitions for the cut locus which seem to fit into the just mentioned distance wave heuristic. Those definitions agree in unbordered manifolds for the cut locus of a point or for the cut locus of a C^2 -smooth submanifold.

With those definitions we get various results valid also for unbordered manifolds and some of them seem to be new even in that case, see e.g. theorem 3.1 and theorem 4.1. However, despite of those surprising analogies, partly caused by the common distance wave concept, described above, the investigation of geodesics and cut loci in bordered manifolds turns out to be much harder than in the unbordered case.

"Classical differential equation techniques cannot suffice for the obstacle problem. No matter how smooth the obstacle, we cannot assume the geodesics are C^2 ; they are not in general, governed by a second order differential equation with Lipschitz conditions. Moreover, each geodesic is the union not only of boundary and interior segments, but also of a set of points which lie on no non trivial boundary or interior segment. This set can have positive measure." 1)

The main reason for these difficulties here is that we may have branching of geodesics, when they meet the boundary. In particular just because of that branching behaviour geo-

1) Quoted from [5].

desics are in general not globally determined by their initial direction. Therefore as already mentioned we have here in general no exponential map of a point oppose to the situation in unbordered manifolds. In addition there occur new phenomena. For instance the cut locus relation for any two points need not to be any longer symmetric, see § 3 p. 50 . The topological classification of the complement of the cut locus C_p of some point p in a compact manifold with dimension larger than two seems to be a difficult problem. In the following part of this introduction we explain the structure of this paper and describe our main results.

In § 2 we present some basic results concerning existence and regularity of geodesics in bordered manifolds. Those results are used in the other paragraphs of this paper.

In § 3 we formulate definitions for cut loci, prove regularity properties of the distance function and give related characterisations for cut loci. Those characterisations hold under weak regularity assumptions for the Riemannian metric and for the boundary ∂M of M . They include as special cases known characterisations of the cut locus of a point in an unbordered manifold. We give now a summary of the results in § 3.

Let A be a closed subset of a space (M,d) of type (Γ) . In order to characterize cut loci we introduce:

Definition 3.1: A point $q \in M \setminus \partial M$ is called extender (relative to A) if there exists a non trivial minimal join from A to q which can be extended minimally beyond q.

Definition 3.1': A point $q \in (M \setminus \partial M)$ is called non-extender (relative to A) if there exists a minimal join from A to q which cannot be extended minimally beyond q.

The statements in definition 3.1 and definition 3.1' are mutual negations. Therefore one can add without changing the content that in both definitions the statements there must hold for all minimal joins from A to the point q.

Defintion 3.2: A point $q \in M \setminus \partial M$ is called pica (relative to A) if q has at least two minimal joins to A with distinct tangents at q.

We discuss in § 3 four definitions for the cut locus. The most important one of these definitions is

Definition 3.4.I: The cut locus of some closed set A is the closure of the set of all non-extenders relative to A. We denote this cut locus of A by C_A .

This definition (as well as the other definitions) seem to fit into the above mentioned distance wave heuristic. This is made plausible by the next theorem. ¹⁾

1) See also figure 3.1 on p. 43.

Theorem 3.1: If the Riemannian metric is locally Lipschitz continuous, then the cut locus C_A is the closure of all points relative to A and $M \setminus (\partial M \cup A \cup C_A)$ is the maximal open set in $M \setminus (\partial M \cup A)$ where $d(A, \cdot)$ is C^1 -smooth.

If the Riemannian metric is $C^{1,1}$ -smooth then theorem 3.1 yields :

Corollary 3.2: If $\partial M = \emptyset$ then $M \setminus C_A$ is the maximal open set of points with a unique minimal join to A and equivalently C_A is closure of all points with at least two minimal joins to A .

In the special case that the set A is a single point, corollary 3.2 gives characterisations of the classical cut locus. This special result has been proved by Bishop in [17], by Klingenberg in [42], by the author in [76] under stronger regularity assumptions for the Riemannian metric. Even in case of a C^∞ -smooth Riemannian metric corollary 3.2 is out of reach for the methods employed in [17], [42], [76] because they all rely essentially on the existence of an (at least continuous) exponential map. Such a map does in general not exist for an arbitrary closed set A in an unbordered complete Riemannian manifold.

We prove in theorem 3.2 a sharpened version of theorem 3.1 for manifolds which are locally C^1 -diffeomorphic to a convex set in \mathbb{R}^n . Further we prove in corollary 3.3 some kind of generalized Gauss Lemma.

In § 3a we prove the Glueing seam Lemma 3.a1 mentioned above on page 14.

In § 4 we study non-extenders. Note we do not use the standard methods of Riemannian geometry like Jacobi fields and comparison theorems.

A classical result for the cut locus C_p of a point p in an unbordered complete C^∞ Riemannian manifold can in our terminology be expressed as follows:

"If a point $q \in C_p$ ¹⁾ is not a pica relative to p , then q must be a conjugate point relative to p ." Hence the point q must be a singular value of the exponential map \exp_p . As already said above in our situation in bordered manifolds an exponential map is in general not available. Nevertheless analysing how minimal geodesics arrive in a neighbourhood of a point q in the cut locus where q is not a pica we find a way to simulate those "singular values" of the exponential map. See in particular (1), (2), and (3) below. Vice versa considering points that are neither conjugates nor picas we are led to the concept of Lipschitz points.

From now on let the underlying Riemannian metric be always C^∞ -smooth.

1) This can also be stated as follows: "If q is a non-extender but not a pica relative to p , then q must be a conjugate point".

Definition 4.1: A point $q \in M \setminus (\partial M \cup A)$ is called a Lipschitz point relative to A if there exists a number L such that in some chart $|\dot{c}_q - \dot{c}_{\bar{q}}| \leq L d(q, \bar{q})$ with $\dot{c}_{\bar{q}}$ the tangent vector at \bar{q} of any normalized minimal joint from \bar{q} to A , $|\cdot|$ being the norm related to the chart.

The subsequent theorem 4.1 is a simplified and weakened version of our main results in § 4.

Theorem 4.1: A point $q \in M \setminus (\partial M \cup A)$ is an extender relative to A iff q is a Lipschitz point relative to A .

This yields immediately with the notation used in definition 4.1: "If there exists a sequence q_n with $\lim q_n = q_0$ such that

$$(1) \quad \lim \frac{d(q_0, q_n)}{|\dot{c}_{q_0} - \dot{c}_{q_n}|} = 0$$

then q_0 is a non-extender.

Crucial in the proof of theorem 4.1 is the following lemma 4.1 which together with (1) above and theorem 4.6 below explains our use of the phrase "Simulation of certain singular values of the exponential map." To formulate lemma 4.1 we need first the subsequent quantitative version of definition 4.1.

Definition 4.2: A point $q \in M \setminus \partial M$ is called ϵ -extender relative to some closed set A if there exists a non-trivial minimal join from A to q which can be extended minimally by length ϵ beyond q with the extension contained in $M \setminus \partial M$.

Lemma 4.1: Let $q_0 \in M \setminus \partial M$ and let $r > 0$ be such that $B_r(q_0)$ is contained in a domain of Riemannian normal coordinates with center q_0 . Then there exists $\epsilon_0 > 0$ and for every $\epsilon \in]0, \epsilon_0[$ a number $\beta > 0$ such that the following holds:

If $A \subset M$ is closed and $q \in B_r(q_0)$ is an ϵ -extender with respect to A then

$$(2) \quad \frac{d(q, \bar{q})}{d(c_q(\epsilon), c_{\bar{q}}(\epsilon))} \geq \beta \quad 1)$$

for all $\bar{q} \in B_{\frac{\epsilon}{2}}(q)$ if $B_{\frac{3\epsilon}{2}}(q) \cap A = \emptyset$.

Theorem 4.6: Let A be any closed subset of M , let q be any point in $M \setminus (A \cup \partial M)$ and let q_n be a sequence of points with $\lim d(q_n, q) = 0$. If there exists some $s_0 > 0$ such that

$$c_q [0, s_0] \subset M \setminus (\partial M \cup A) \quad \text{and}$$

$$(3) \quad \lim \frac{d(q, q_n)}{d(c_q(s_0), c_{q_n}(s_0))} = 0$$

$c_q(s)$, $c_{q_n}(s)$ normalized minimal joins from q , q_n to A , then q must be a non-extender relative to A . Thus q is a non-Lipschitz point by theorem 4.1.

1) We denote by $c_q(t)$, $c_{\bar{q}}(t)$ normalized minimal joins from q , \bar{q} to A respectively.

In § 5 we apply results of § 3 and § 4 in order to derive several theorems. Some of these theorems are well known. However, giving new proofs for these theorems I want to show that the technics used in § 3 and § 4 which we originally developed to investigate cut loci in bordered manifolds give also a common frame for results of Jacobi, Bangert, Federer, Kleinjohann and R. Walter which belong to apparently different topics. Let A be any closed set in (M, d) . The combination of theorem 3.1 and a sharpened version of one direction in theorem 4.1 yield:

Theorem 5.1: The gradient of the distance function $d(A, \cdot)$ is locally Lipschitz continuous on $M \setminus (C_A \cup \partial M \cup A)$.

Using that the Gauss-Bonnet theorem has been proved under weak regularity assumptions cf. [36] p. 343, theorem 5.1 makes it possible to apply the Gauss-Bonnet theorem to pieces of distance hypersurfaces relative to A ²⁾ which do not meet $C_A \cup \partial M \cup A$.

Using theorem 4.1 we give a new proof of a well known theorem i.e. the subsequent theorem 5.2 a special case of which is a famous result of Jacobi, see [16] p. 231. All proofs known to us for theorem 5.2 as well as for Jacobi's theorem use second variation and index form technics, while we do not use these methods here.

1) See in particular [38] 3.7 Satz and cf. theorem 5.7 in this paper.

2) i.e. level surfaces of the distance function $d(A, \cdot)$.

Theorem 5.2: Let A be a C^2 -smooth submanifold of an n -dimensional unbordered Riemannian manifold. Then any geodesic starting (vertically) in A does not minimize the distance to A beyond the first focal point.

In [29] Federer investigates in Euclidean space a class of sets which enjoy the so called unique footpoint property. Any closed set A has the unique footpoint property if there is a neighbourhood $U(A)$ of A such that for every point $q \in U(A)$ there exists a unique point $\xi(q)$ of A closest to q . This map $\xi : U(A) \rightarrow A$ is called metric projection. Federer calls sets with unique footpoint property sets of positive reach. Prior to [29] a similar concept had been studied by Durand [27]. Bangert, Kleinjohann und R. Walter investigate sets of positive reach in Riemannian manifolds, see [11], [39], [38], [71]. Following Bangert and Kleinjohann we call a set with local unique footpoint property shortly "EFP-set". All convex sets and all sets with C^2 -smooth boundary belong to this class, see [11]. In Riemannian geometry EFP-sets are important for the investigation of convex sets, see [72] and [39]. Using the concept of cut loci we give now a simple characterisation of EFP-sets.

Theorem 5.7: A closed set A in an unbordered complete Riemannian manifold is an EFP-set iff there exists a neighbourhood U of A such that U does not meet the cut locus C_A of A .

Using a result of V. Bangert [11] we find:

Corollary 5.9: If a set avoids locally its cut locus then this property is invariant under $C^{1,1}$ -diffeomorphisms and is independent of the Riemannian structure.

Further we get the following result of Federer and R. Walter the most general version of which is due to Kleinjohann [39]:

Corollary 5.10: If A is an EFP-set in an unbordered Riemannian manifold, then there exists an open set U containing A such that the metric projection $\xi : U \setminus A \rightarrow A$ is locally Lipschitz continuous.

The subsequent theorem being the last result in § 5 describes mainly for a certain class of manifolds some simple relations between the number of isolated points in the cut locus and topological properties of the related manifold.

Theorem 5.11: Let A be a closed bordered n -dimensional C^2 -smooth submanifold of an unbordered n -dimensional complete Riemannian manifold M . Let $N_A := \{q \in M \setminus A / d(A, \cdot) \text{ not differentiable in } q\}$ and denote by J_{N_A} the set of isolated points in N_A . Let $|J_{N_A}|$ be the number of points in J_{N_A} and let $k \in \mathbb{N} \cup \{\infty\}$ ¹⁾ be the number of connected components of ∂A . Then the following statements are valid:

a) We have $k \geq |J_{N_A}|$.

1) \mathbb{N} denotes the set of natural numbers.

- b) Let us assume now that k is finite. If $|J_{N_A}| \geq k$ then $M \setminus A$ is diffeomorphic to the union of k disjoint open unit discs and ∂A is diffeomorphic to the union of k disjoint unit spheres.
- c) If $N_A = \emptyset$, then $(M \setminus A) \cup \partial A$ is diffeomorphic to the exterior normal bundle over A .
- d) If A is a single point p and if $|J_{N_p}| \geq 1$ then M is homeomorphic to the n -dimensional unit-sphere.

As an illustration of the results and methods of §§ 2, 3, 4 we study in § 6 the cut locus of special bordered surfaces.

Let \hat{M} be an unbordered, complete simply connected two-dimensional Riemannian manifold without conjugate points.¹⁾ We call such a manifold \hat{M} a space of type (*). Let S be a closed topological subsurface of a space of type (*) and assume ∂S contains only locally rectifiable curves. Then we call S a space of type (**).

Theorem 6.1: Let S be a simply connected subset in a space of type (*). Then any two points of S can be joined by at most one shortest normalized path contained in S .

We have the following characterisation of a large class of simply connected bordered subsurfaces in a space of type (*).

1) In other words \hat{M} is diffeomorphic to R^2 and \hat{M} is complete and has no conjugate points.

Theorem 6.2: Let S be a space of type (**). Then the following statements are all equivalent :

- a) The subsurface S is simply connected.
- b) There exists a point p in S with $C_p \setminus \partial S = \emptyset$, C_p the cut locus in S of the point p .
- c) For all points p in S is $C_p = \emptyset$.
- d) There exists a point p in S such that the distance function $d(p, \cdot)$ is C^1 -smooth on $S \setminus (\partial S \cup \{p\})$.
- e) For all points p in S is $d(p, \cdot)$ C^1 -smooth on $S \setminus (\partial S \cup \{p\})$ and has a locally Lipschitz continuous gradient there.
- f) Any two points of S can be joined by exactly one shortest normalized path contained in S .
- g) Any two points of $S \setminus \partial S$ can be joined by exactly one shortest normalized path contained in S .

Theorem 6.3 gives a detailed description of the cut locus $C_{\{p,q\}}$ of two distinct points p, q in a simply connected space S of type (**). Some part of that result can be described as follows:

The cut locus $C_{\{p,q\}}$ is a one-dimensional topological submanifold of S and $C_{\{p,q\}}$ is homeomorphic to one of the following intervals $]0,1[$, $[0,1]$, $[0,1[$. Further $C_{\{p,q\}} \setminus \partial S$ is a C^1 -smooth submanifold of M and $C_{\{p,q\}} \setminus \partial S$ is C^1 -diffeomorphic to \mathbb{R} ¹⁾. A point $x \in S$ is contained in $C_{\{p,q\}}$ if there exist two minimal joins g_p, g_q going from x to p, q respectively with $g_p \cap g_q = \{x\}$ and length $g_p = \text{length } g_q$. At least in case S is compact the cut locus $C_{\{p,q\}}$ separates S ²⁾ i.e. $S \setminus C_{\{p,q\}}$ has two components K_p, K_q , $p \in K_p$, $q \in K_q$. Both components K_p, K_q are simply connected ²⁾.

1) \mathbb{R} denotes the real numbers.

2) Cf. remark 6.9.

Let S be a closed bordered subsurface of an unbordered, simply connected, complete, two-dimensional Riemannian manifold M , M having nowhere positive curvature ¹⁾. We assume that ∂S consists of locally rectifiable boundary curves. Using theorem 6.3 we prove theorem 6.5 which gives a detailed description of the cut locus C_p for an arbitrary point p in the above space S , if S is homotopic to a circle. Some part of theorem 6.5 says:

There exists a continuous embedding

$$\bar{\Psi} : J \rightarrow S \quad , \quad J \in \{[0,1], [0,1[\} ,$$

$$\text{with } \bar{\Psi}(J) = C_p , \quad \bar{\Psi}(]0,1[) = C_p \cap S \setminus \partial S ,$$

the restriction

$$\bar{\Psi} / :]0,1[\rightarrow S$$

is a C^1 -smooth embedding. The point $\bar{\Psi}(0)$ belongs to the frontier of the bounded component of $M \setminus S$.

We used the above results theorem 6.1, 6.2, 6.3, 6.5 as basic tools to attack our subsequent conjecture which we could not prove completely up to now.

1) In other words M is diffeomorphic to Euclidean plane, M is complete and the curvature on M is everywhere smaller than or equal to zero.

Conjecture: Let S be a space of type (**), then we have:

- a) For all points $p \in S$ the cut locus C_p is a contractible (may be disconnected) tree.
- b) $(\text{Rk } H_1(S))^{1)} = 1) \Leftrightarrow$ (There exists a point $p_0 \in S$ such that $C_{p_0} \setminus \partial S$ is C^1 -diffeomorphic to R^1)
 \Leftrightarrow (For all $p \in S$ $C_p \setminus \partial S$ is C^1 -diffeomorphic to R^1)
- c) $(\text{Rk } H_1(S) = n) \Leftrightarrow$ (For all points $p \in S$ $n =$ number of ends of $C_p \setminus \partial M - \text{Rk } H_0(C_p \setminus \partial S)$) \Leftrightarrow (There exists a point $p_0 \in S$ such that $n =$ number of ends of $C_{p_0} \setminus \partial S - \text{Rk } H_0(C_{p_0} \setminus \partial S)$). 2)

We want to make a final remark on a result of particular interest that we shall mention only here in the introduction. The authors prove in [5] local bipoint uniqueness for geodesics in bordered Riemannian manifolds with C^∞ -smooth boundary. This means every point there has a neighbourhood in which every pair of points can be joined by a unique (normalized) minimal join, cf. theorem 5.6.

Using this result, then the result, the theorem Existenzsatz 16 in Rinow's book [59], p. 277, 278 holds for compact non

1) $\text{Rk } H_1$ is the rank of the first homology group of S .

2) The deeper reason why the above conjecture should be true is as follows: Namely we think that for a space of type (***) roughly spoken the following holds: In the universal covering space $(\tilde{\eta}, \tilde{S})$ of S the covering of the cut locus C_p bounds fundamental domains. Further, pick in S the minimal loops (with base point p) which generate the fundamental group of S . Lift those loops to \tilde{S} . Start the lifts in a point $p_1 \in \tilde{\eta}^{-1}(p) \in \tilde{S}$ and call the end points of the lifts p_2, \dots, p_k . Then the cut locus of $\{p_1, \dots, p_k\}$ (is contained in a fundamental domain and) covers C_p .

contractible Riemannian manifolds with C^∞ -smooth boundary. Thus one gets in particular the following result.

Theorem: Let M be a non-contractible, compact Riemannian manifold with C^∞ -smooth boundary. Then there exists in M a non-trivial closed C^1 -smooth geodesic.

The latter result has been proved by us in [77] for non simply connected, bordered, compact Riemannian manifolds under weaker assumptions for the Riemannian metric and for the boundary.

In 1978 we started to investigate shortest paths, distance geometry and cut loci in bordered Riemannian manifolds. We did not know in those days that at the University of Illinois S. and R. Alexander, R.L. Bishop and J.D. Berg were already active in this field. Since those days there have come many important contributions to this field, see [2], [3], [4], [5], [12].

In a series of recent papers V. Arnold in Moscow has considered the obstacle problem, see [6], [7], [8], [9]. ¹⁾

V. Arnold is carrying out a general program which identifies standard singularities related to the geometry of

1) We quote now from [5].

groups generated by reflections with normal forms for singularities occurring in variational problems. This investigation leads him to an analysis of wavefronts ¹⁾ around an obstacle in general position in Euclidean space. However, as Arnold says this investigation is far from being complete even in Euclidean 3-space, see [6], [10] p. 65.

S. Alexander, J.D. Berg and R.L. Bishop investigate in [5] the local distance geometry in a Riemannian manifold with smooth boundary. Their emphasis is on the structure of fields of geodesics. ²⁾ In the presence of an obstacle the description of such fields in terms of differential equations is no longer feasible; as an alternative they present in [5] a differential inequality which functions as a one-sided version of the Jacobi equation. In consequence they obtain a local bipoint uniqueness for geodesics and a geometric estimate on the distance below which bipoint uniqueness holds. ³⁾ They give a statement of regularity for geodesics involving a decomposition into tangential and normal part. Specifically, the tangential part is smoother by one degree than the geodesic itself which is $C^{1,1}$ -smooth and the normal part satisfies a convexity condition.

1) Those wavefronts are level surfaces of the intrinsic distance function.

2) We quote from [5].

3) They prove: Every point has a neighbourhood U such that for every p, q in U there is a unique minimal geodesic segment joining p and q , and there is no other geodesic segment joining p and q and lying in U .

They show the existence and continuity of Jacobi fields and apply this to get results which contribute to understand the local bifurcation behaviour of geodesics in bordered manifolds. The latter will be of essential importance for further global investigations of geodesics and distance geometry in bordered manifolds. Investigating fields of geodesics with a common initial tangent vector S. Alexander, J. Berg and R. Bishop are led to the following conjecture:

"Every boundary point p has a neighbourhood in which two geodesics coincide if they have the same initial tangent vector and length and if their endpoints lie on the boundary." They prove in [5] that this is equivalent with the following conjecture, which they call "Cauchy uniqueness for manifolds with boundary".

Every boundary point has a neighbourhood U such that for any two geodesics in U with the same initial tangent vector and length, one of the geodesics is a lift of the other." Here a lift $\tilde{\gamma}$ of a normalized geodesic $\gamma : [0,1] \rightarrow M$ is a normalized C^1 -smooth curve which has the same length and initial tangent vector as γ ; further $\tilde{\gamma}$ consists of an initial segment of γ say $\tilde{\gamma}[0,u] = \gamma[0,u]$, $u \in [0,1]$ and $\tilde{\gamma}[u,1]$ is a geodesic relative to the ambient unbordered Riemannian manifold $\hat{M} \supset M$. It is shown in [5] that for small enough neighbourhoods those lifts yield geodesics contained in M . The lift endpoints trace out the involute curve σ of γ namely

$$\sigma(u) = \exp(1-u) \dot{\gamma}(u), \quad 0 \leq u \leq 1$$

\exp being the exponential map of \hat{M} . Therefore the validity of the above conjectures would imply that locally every initial vector and length determines exactly one involute and those involutes describe all possible bifurcations.

In 1980 a group of analysts in Pisa including E. De Giorgi, A. Marino, D. Scolozzi and M. Tosques became interested into problems for geodesics in bordered n -dimensional submanifolds of \mathbb{R}^n (geodetiche con ostacolo). The geodesics are interpreted as stationary points of the energy-functional. One has for those stationary points unilateral constraints arising from the condition that the geodesic must stay in the bordered manifold. Those analysts in Pisa apply a theory of functionals, not necessarily C^2 or convex on infinite-dimensional spaces which was initiated by De Giorgi, Marino and Tosques see [24]. Marino and Scolozzi [48] have shown that geodesics have Lipschitz continuous derivatives and that for a large class of n -dimensional bordered submanifolds ¹⁾ of \mathbb{R}^n there exist infinitely many geodesics (the supremum of whose lengths is infinite) joining two given points in the submanifold. ²⁾ Using different methods than in [5] D. Scolozzi [62] has proved bipoint uniqueness for geodesics. ³⁾ D. Scolozzi has also proved the existence of closed geodesics for certain bordered n -dimensional submanifolds of \mathbb{R}^n .

- 1) For instance if the submanifold is complement of an open set O , O diffeomorphic to the open n -dimensional unit disc.
- 2) They apply here Lusternik-Schnirelmann category theory.
- 3) Scolozzi uses analytical and variational methods.

Of course it is important that a perhaps new or young concept contributes to solve classical problems in its field, here in differential geometry. As far as we know there exist at least two places where intrinsic considerations in bordered manifolds have been used in the proofs of major theorems. Those are the theorem of Efimov ¹⁾ and the theorem of Cohn Vossen the latter being the generalisation of the Gauss Bonnet theorem to complete surfaces. It is the main problem in the long and difficult proof of Efimov's theorem [28], [43] to show that certain bordered surfaces are not complete as metric spaces in their interior metric. This proof also uses the regularity of geodesics in those bordered surfaces, see [43], p. 537-541. One of the most promising fields for applications of distance geometry in bordered manifolds may perhaps be seen in the investigation of diffusion processes in bordered manifolds. See [54] § 4 and see the list of problems in [54] p. 28. Those investigations have applications in the spectral geometry of bordered manifolds, see [54].

1) Efimov's theorem says: "No surface can be C^2 immersed in Euclidean 3-space so as to be complete in the induced Riemannian metric, with Gauss curvature $K \leq \text{const} < 0$."

§ 2 Preliminaries

In this paragraph we state and explain some results from [77] concerning existence and regularity of geodesics in bordered manifolds. These results are used many times in this paper. Crucial for our work is the concept of interior metric in the sense of Rinow, see [59] p.121 and [77] p. 1 . In a path connected metric space (M,d) we can define the length of a path $c(t)$, $c(t): I \rightarrow (M,d)$ as the supremum of sums of distances between partition points taken over all finite partitions of the parameter interval I . If this supremum denoted by $L_d(c)$ is finite then the path c is called rectifiable. Now if any two points x,y in (M,d) can be joined by a rectifiable path then one can define a new metric $\tilde{d}(\ , \)$ on M , by taking $\tilde{d}(x,y)$ as infimum of lengths for all rectifiable paths joining x and y . If the metrics $d(\ , \)$ and $\tilde{d}(\ , \)$ agree we say (M,d) is a space with an interior metric. Very important for us is the following result of Rinow.

Lemma 2.1: In a locally compact and complete metric space (M,d) with an interior metric: any two points x,y can be joined by a distance realizing path c , thus $L_d(c) = d(x,y)$.

For a short proof of lemma 2.1 see [77] p.8, see also [59]p. 141.

The following theorem is essentially one of the main results in [77], see also remark 2.1 below.¹⁾

1) The results given in theorem 2.1a,b below seem to be more or less well known meanwhile. Now there exist various recent papers giving results which are overlapping with the subsequent theorem 2.1a,b, see [4], [5], [12], [48], for theorem 2.1b, see also [43], p. 536. Moreover we realized that regularity problems closely related to theorem 2.1b have also been treated in older papers, [64], [22], [35]. The existence problem (theorem 2.1a) can even be traced back to Hilbert [36].

Theorem 2.1: Let M be an n -dimensional submanifold of an n -dimensional C^1 -smooth manifold \widehat{M}, \widehat{M} carrying a locally Lipschitz continuous Riemannian metric (g_{ij}) . Let the boundary ∂M of M be an $(n-1)$ -dimensional topological submanifold of M , ∂M may be empty. Defining the distance $d(\cdot, \cdot)$ on M by $d(p,q) := \inf\{\text{length } c \mid c \text{ a rectifiable path in } M \text{ joining } p \text{ and } q\}$, we require (M,d) to be a complete, locally compact metric space. Under these assumptions (M,d) is a space with an interior metric. We call the just described space a space of type (Δ) .

We have

Theorem 2.1.a: Any two points p,q in (M,d) can be joined within of M by a distance realizing path c , i.e. $L_d(c) = d(p,q)$.

Theorem 2.1.b: If M is in addition locally C^1 -diffeomorphic to convex sets in \mathbb{R}^n , then geodesics in (M,d) i.e. locally shortest paths in (M,d) , are C^1 -smooth, if they are normalized. Let (ϕ, U) be a chart of $M \supset U$, $\phi(U) = K$ a compact, convex set in $(\mathbb{R}^n, |\cdot|)$. Then we have a constant number R such that for any normalized minimal join $c(t)$ contained in U ,

$$|\dot{c}(t) - \dot{c}(\bar{t})| \leq R|t - \bar{t}|^{1/2}, \quad \dot{c}(t) \text{ denoting the tangent vector,}$$

where c abbreviates $\phi \circ c$. 1)

Remark 2.1: The Riemannian metric (g_{ij}) in the assumption of theorem 2.1 induces a distance function $\hat{d}(\cdot, \cdot)$ on M and

1) Note we want to keep our notations simple and short. Therefore we shall often use the same notations for objects (i.e. paths, vectors) and their related representations under some chart. We hope that the meaning will be always clear from the context. Note $\phi \circ c$ is a $C^{1+1/2}$ -smooth path in $K \subset \mathbb{R}^n$, in the manifold structure \hat{c} is here only C^1 -smooth.

(M, \bar{d}) is easily shown to be a metric space with interior metric, see [77] p. 25. We require in the assumption of theorem 2.1 that any two points in M can be joined by a rectifiable path c , c contained in M . Clearly initially we mean here that c is rectifiable relative to the already existing metric $\bar{d}(\cdot, \cdot)$, i.e. $L_{\bar{d}}(c) < \infty$. However it follows by a straightforward consideration immediately from the definition that we have here for any path $c: I \rightarrow (M, d)$ the equality $L_{\bar{d}}(c) = L_d(c)$. Therefore it is almost trivial, see [77] p.28, that (M, d) in theorem 2.1 is a space with an interior metric. Therefore and because (M, d) is locally compact and complete theorem 2.1a follows by Lemma 2.1.

Remark 2.2: Let M be a closed n -dimensional submanifold of an n -dimensional complete Riemannian manifold \bar{M} , ∂M an $(n-1)$ -dimensional topological submanifold of \bar{M} . Then M is a space of type (Δ) if ∂M can be locally (i.e. in some chart) represented as a graph of a continuous function. This case includes in particular the situation where M is locally C^1 -diffeomorphic to convex sets in \mathbb{R}^n . Of course all this covers the special case where ∂M is a C^k -smooth ($k \geq 1$) manifold. The proposition 6.1 in § 6 gives a class of examples, where the assumptions of theorem 2.1a are fulfilled while ∂M need not to be locally the graph of a continuous function.

Remark 2.3: Let (M, d) be a space of type (Δ) . We do not know whether (M, d) is homeomorphic to the submanifold M of \hat{M} .

Let us assume throughout the whole paper unless we say anything different that the considered metric spaces (M,d) and all those spaces where our considerations take place are at least locally compact and complete metric spaces with an interior metric.

Now we give two minor results which will turn out to be useful later on. The following result is well known. Therefore we are not pedantic in the formulation of it's statement and we omit it's proof. See e.g. [20] p.24 for a proof of Assertion 2.1: If a sequence of minimal joins is contained in some bounded set, then this sequence contains a subsequence which converges uniformly against a minimal join.

The next assertion does in general not hold in an arbitrary metric space, even if this space is complete and homeomorphic to \mathbb{R}^1 , see e.g. [77] p.4.

Assertion 2.2: Let A be a closed subset of (M,d) and let q be any point in M . Then there exists a point $\tilde{q} \in A$ such that $d(q,\tilde{q}) = d(q,A)$, with $d(q,A) := \inf\{d(q,\tilde{q}) | \tilde{q} \in A\}$.

Proof: Let α be any real number with $d(q,A) < \alpha$. The set $\tilde{A} := \{x \in M, d(x,q) \leq \alpha\}$ is obviously a non empty, bounded and closed subset of (M,d) . Therefore \tilde{A} is compact, since our metric space (M,d) enjoys the Heine-Borel property, see e.g. [77] p.2. The compactness of \tilde{A} guarantees the existence of a point $\tilde{q} \in \tilde{A} \cap A$ with $d(q,\tilde{q}) = d(q,\tilde{A})$. This also proves our claim $d(\tilde{q},q) = d(\tilde{q},A)$, because we have $d(y,q) > d(\tilde{q},q)$ for

all points $y \in (A \setminus \tilde{A})$ by definition of \tilde{A} . In case $A \setminus \tilde{A} = \emptyset$ we get here $A = \tilde{A}$ and are finished too.

§ 3 Extenders, Picas, Cut loci and regularity of the distance-function

In this paragraph we discuss the problem "how to formulate a definition for cut loci", prove regularity properties of the distance-function and give related characterisations of cut loci. Those characterisations hold under weak regularity assumptions and include as special cases known characterisations of cut loci relative to a point in unbordered manifolds as given in [76], [43] p.134, [17], see corollary 3.2. During the whole paragraph, unless we say anything else, let (M,d) always be a space as given in the assumptions of theorem 2.1 with M submanifold of the Riemannian manifold (\hat{M},g) and let A be a closed subset of (M,d) . - We introduce the following definitions for the technical reason of convenience, but we also think that they describe crucial properties.

Definition 3.1: A point $q \in (M \setminus \partial M)$ is called extender relative to A or extender in short if there exists a non trivial minimal join from A to q which can be extended minimally beyond q .

Definition 3.1': A point $q \in (M \setminus \partial M)$ is called non-extender relative to A or non-extender in short if there exists a non trivial minimal join from A to q which cannot be extended minimally beyond q .

Assertion 3.1: Although this is not literally stated it is easily seen that the statements in definition 3.1 and defini-

tion 3.1.' are mutual negations. Therefore one can add without changing the content that in both definitions 3.1 and 3.1' the statements there must hold for all minimal joins from A to the point q .

Definition 3.2: A point $q \in (M \setminus \partial M)$ is called a pica relative to A or pica in short if q has at least two minimal joins to A with distinct tangents at q .

A pica is clearly a non-extender however not every non-extender is a pica, even not in an unbordered manifold with a real analytic Riemannian metric if we consider picas and non-extendors relative to one point sets. The next paragraph § 4 contains a detailed investigation of the relation between picas and non-extendors.

Somehow typical for bordered manifolds is the existence of the below defined branching points, which are necessarily located on the boundary ∂M if the Riemannian metric is $C^{1,1}$ -smooth. However the following definition does not exclude branching points which are located in $(M \setminus \partial M)$ and such may perhaps occur in case the Riemannian metric is only Lipschitz continuous.

Definition 3.3: A point $b \in M$ is called branching point relative to some closed set A , $b \notin A$, if there exists a non stationary sequence of points (q_n) , $n \in \mathbb{N}$, such that for some positive

real number s_b , $c_{q_n}(s_b) = b$ for all $n \in \mathbb{N}$ and $\bigcup_{n \in \mathbb{N}} c_{q_n}[0, s_b] = \{b\}$,

with $s \mapsto c_{q_n}(s)$ a normalized minimal join from q_n to the set A .

It is easy to give examples for branching points. In figure 3.1 as well as in figure 3.4 below the point q_2 is a branching point relative to the set $\{p\}$.

The following heuristical discussion, see also [2], together with the subsequent results in this paragraph and results in § 6 may help to motivate our definitions 3.4.I, 3.4.II, 3.4.III, 3.4.IV below proposed for cut loci. One seeks a formulation of the classical notion of the cut locus which is appropriate to the setting of bordered manifolds. One might reasonably wish that say respective some point p the cut locus C_p should be a closed set of measure zero. Further $M \setminus C_p$ should be contractible or at least have a fairly simple homotopy type. By this we mean that $M \setminus C_p$ should be contractible or at least highly connected.

Probably all these requirements can be met only under stronger regularity conditions and only in more special situations than those we need for our results in this paragraph. Intuitively for say physical reasons it seems to us that the distance wave concept mentioned in the introduction fulfills in a large class of two dimensional manifolds the above stated requirements for the cut locus C_p relative to a point p . Namely at least for a large class of two dimensional manifolds the following statements should hold. First the cut locus C_p in the distance wave concept being the locus where the

distance wave fronts relative to p have self interference seems to have measure zero on the Riemannian manifold (\hat{M}, g) . Further viewing $M \setminus C_p$ as being somehow backward exhausted by the distance wave fronts relative to the point p it seems plausible that $M \setminus C_p$ is contractible in p . However the investigation of the distance wave concept for cut loci might be justified also on grounds of it's own geometric appeal.

Now we discuss the problem of defining cut loci at the example of a bordered manifold described in figure 3.1 below. This figure describes a compact subspace (M, d) of the Euclidean plane. Here (M, d) is topologically a closed annulus whose boundary is given by the circle and the exterior simple closed curve in figure 3.1. Even this primitive example will already exhibit problems and phenomena which do not occur in unbordered manifolds, but are typical in our setting.

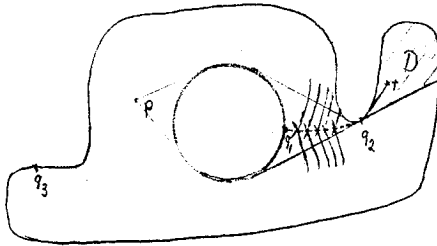


Figure 3.1

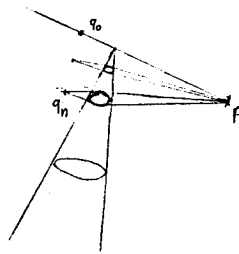


Figure 3.2

Let us take the classical definition of cut loci literally in the situation of the example in figure 3.1. Then the cut locus relative to the point p is here the set of all non-extendors relative the point p in sense of definition 3.1. However we wish to include also non-extendors which are located on the boundary. With this proposed definition we immediately run into difficulties. First it is not clear how to apply this definition intuitively reasonable to points which are located on the boundary say if a minimal join meets ∂M transversally. Moreover we also don't seem to get what we want by the literal application of this definition in cases where a minimal join starting in p meets ∂M tangentially as say in case of the points q_2 and q_3 in figure 3.1. Here it can be arranged with C^∞ -smooth boundary that q_3 is a non-extendor while q_2 is an extender, since minimal joins from p to q_2 can be prolonged minimally up to all points located

in the shaded set D , since the point q_2 is a branching point relative to the point p . In this example the cut locus C_p would contain the isolated point q_3 and the dotted arc between q_1 and q_2 except the point q_2 . Therefore already in this example we observe a phenomenon which does not occur in unbordered complete Riemannian manifolds, i.e. 'the set of non-extendors relative to a point p , i.e. here the half open dotted arc, has a cluster point q_2 being an extender'. In the example in figure 3.1 this cluster point is located on the boundary. However, in figure 3.2 we have the situation that a sequence q_n of non-extendors relative to the point p is converging against an extender q_0 , where the point q_0 is not on the boundary. Here figure 3.2 describes a bordered submanifold of Euclidean 3-space, where M is the closed non-convex set being the complement of the open convex cone scetched in the drawing.

If we take instead of a single point some closed set A , we even may have in an unbordered complete manifold M the situation that a sequence q_n of non-extendors relative to A is converging against an extender $q_0 \in (M \setminus A)$. This situation occurs in figure 3.3 where M is the Euclidean 3-space and the set A is given by the two non-dotted rays emanating from the point p . Here the points $q_n, n \in \mathbb{N}$ are not contained in the plane spanned by the two rays in A .



Figure 3.3

In [76] we gave in a complete unbordered C^∞ -smooth Riemannian manifold a characterisation of the cut locus for any point p as closure of all points having at least two distinct minimal joins to p . Taking this characterisation as definition then one can easily show that say for a space (M,d) as assumed in theorem 2.1 the complement of the cut locus is contractible. However, with this definition for the cut locus the shaded set D in figure 3.1 would contain subsets of the cut locus relative to p which are open in $(M \setminus \partial M)$. Therefore using this definition the cut locus relative to some point would in general not have measure zero not even in a compact submanifold M of the Euclidean plane with real analytic boundary.

We return now to the example in figure 3.1. Clearly with the classical definition literally applied, the cut locus here is not closed since e.g. the cluster point q_2 is missing. Further it would contradict our intuition that here e.g.

the point q_3 should belong to the cut locus, since q_3 does not belong to the closure of points where the distance wave front relative to some source point p has self-interference, which in this example happens to be on the dotted arc between q_2 and q_3 , see figure 3.1.

Although the main objective of this paper is the investigation of cut loci of points, we give four definitions of the cut locus of some closed set in a bordered manifold. We define cut loci of closed sets because the results in this paragraph and various results in other sections hold for cut loci of general closed sets.

Definition 3.4.I: The cut locus of some closed set A is the closure of the set of all non-extenders relative to A . We denote this cut locus of A by C_A^I .

Definition 3.4.II: The cut locus of some closed set A is the closure of the set of all picas relative to A . We denote this cut locus of A by C_A^{II} .

Definition 3.4.III: The cut locus of some closed set A is the closure of all points (possibly on the boundary) where at least two minimal joins starting in A end up with distinct tangents. We denote this cut locus by C_A^{III} .

Definition 3.4.IV: The cut locus of some closed set A in a manifold M is the set of all non-extenders relative to A together with its clusterpoints on ∂M . We denote this cut locus by C_A^{IV} .

Remark: a) In unbordered complete n-dimensional Riemannian manifolds all four definitions for cut loci are equivalent if the related closed set A is a point or a C^2 -smooth submanifold or if e.g. A is a bordered submanifold with C^2 -smooth (n-1)dimensional boundary.

b) Further we think those four definitions agree for the cut locus of a point in a bordered two-dimensional manifold M, M being submanifold of a 2-dimensional unbordered complete C^∞ -smooth Riemannian manifold \hat{M} with \hat{M} having curvature everywhere smaller or equal to zero.

We will return to this remark in paragraph 6.

Remark: In general we have that definition 3.4.I and 3.4.II are equivalent by theorem 3.1a, thus $C_A^I = C_A^{II}$. It is obvious that for closed sets in unbordered manifolds $C_A^{IV} \subset C_A^I$, see in particular remark 4.2. We obviously have $C_A^{III} \supset C_A^{II}$, and $(C_A^{III} \setminus \partial M) = (C_A^{II} \setminus \partial M)$ by definition, but it is possible that $(C_A^{III} \setminus C_A^{II}) \neq \emptyset$. Take as an example for (M,d) a closed hemisphere of the twodimensional unitsphere. Then for a point p on ∂M we have that C_p^{III} consists of the point located diametrically to p on the boundary circle of M, while C_p^{II} is empty. Therefore we have in general the following inclusions $C_A^{IV} \subset C_A^I = C_A^{II} \subset C_A^{III}$ where e.g. the case $C_A^{IV} \subsetneq C_A^I = C_A^{II} \subsetneq C_A^{III}$ may happen. We obviously get from these considerations that $C_A^I \setminus \partial M = C_A^{II} \setminus \partial M = C_A^{III} \setminus \partial M$. However, it may happen that $C_A^{IV} \setminus \partial M \subsetneq C_A^I \setminus M$ cf. remark 4.2. Therefore it would be interesting to know: "If A is a point p, under what additional assumptions (if they are necessary) holds $C_p^{IV} \setminus \partial M = C_p^I \setminus \partial M$."

Remark: In an unbordered complete C^∞ -Riemannian manifold M the cut locus relation is symmetric relative any point p . This means if a point q belongs to the cut locus C_p of the point p then p belongs to C_q . It is of particular interest for our work that in a bordered Riemannian manifold this symmetry does not hold in general, so e.g. in figure 3.1 the point $r \in D \cap (M \setminus \partial M)$ does not belong to $C_p^I \cup C_p^{II} \cup C_p^{III} \cup C_p^{IV}$ while we have $p \in C_p^I \cap C_p^{II} \cap C_p^{III} \cap C_p^{IV}$. We mention that the above symmetry in the cut locus relation does in general not hold for a closed subset A in an unbordered complete Riemannian manifold. Take e.g. as set A the unit circle in the Euclidean plane E^2 . Then the cut locus of the set A in E^2 is the midpoint e of the unit disc, while the cut locus of the point e is empty.

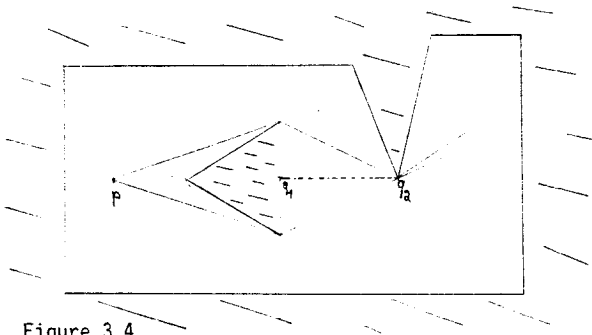


Figure 3.4

Remark: The example in figure 3.4 shows a phenomenon, which does not occur for cut loci in unbordered manifolds even not for the cut locus of some closed set. Figure 3.4 describes a bordered compact submanifold M of the Euclidean plane. Here M has the topological type of a closed annulus, M is described by the exterior polygon and by the interior equilateral triangle.¹⁾ The cut locus relative to the point p in the sense of all four definitions is the dotted arc between the points q_1 and q_2 . Now if we apply in a generalising way the definitions of pica and extender literally also to the boundary point q_2 then we have that the point q_2 is a (generalised) pica relative to p and at the same time a (generalised) extender relative to p .

1) The complement of M is shaded.

As already mentioned above, the following theorem 3.1 and in particular the statement in theorem 3.1b may be seen as a motivation of our definitions 3.4.I, 3.4.II, 3.4.III for cut loci in relation to the distance wave concept.

Theorem 3.1: Let (M,d) be a space as given in the assumption of theorem 2.1, A a closed subset of (M,d) . Then the following statements hold.

3.1.a The closure of all nonextenders relative to A , i.e. the cut locus C_A^I equals the closure of all picas relative to A thus $C_A^I = \overline{C_A^{II}}$. Equivalently, $M \setminus (\partial M \cup C_A^I)$ is the maximal open set of points in $M \setminus (\partial M \cup A)$ where minimal joins to A start with a unique common initial direction.

3.1.b The set $M \setminus (\partial M \cup C_A^I)$ equals the maximal open set of points in $M \setminus (\partial M \cup A)$ where the function $d(A, \cdot)$ is C^1 -smooth.

Theorem 3.1.a yields immediately the subsequent corollaries.

Corollary 3.1: We have $C_A^I \setminus \partial M = \overline{C_A^{III}} \setminus \partial M$.

Corollary 3.1': The set of picas relative to A is dense in the set of non-extenders relative to A .

For unbordered manifolds theorem 3.1a gives directly

Corollary 3.2: If $\partial M = \emptyset$, M a complete C^2 -smooth manifold, where the Riemannian metric (g_{ij}) has in local coordinates locally Lipschitz continuous derivatives, then $M \setminus C_A^I$ is the maximal open set in M of points with unique (nontrivial) minimal join to A . Equivalently, the closure of non-extenders relative to A , i.e. the set C_A^I equals the closure of points with at least two distinct minimal joins to A .

Remark: In the special case that the set A is a point, corollary 3.2 gives characterisations of the classical cut locus. This special results has been proved in [17], [42], p. 134, [76] under stronger regularity assumptions for the Riemannian metric. Vice versa, even in case of a C^∞ -smooth Riemannian metric corollary 3.2 is out of reach for the methods employed in [17], [42], [76] since all those methods rely essentially on the existence of an (at least continuous) exponential map, which in general does not exist for arbitrary closed sets in an unbordered complete Riemannian manifold.

We denote the closure of all picas relative to A by $(\text{Picas})\bar{\setminus} aM$ and the closure of all non-extenders relative to A by $(\text{Non-extenders})\bar{\setminus} aM$.

Proof of theorem 3.1 : A short look on the definitions tells that it is sufficient to show the equality

$(\text{Picas})\bar{\setminus} aM = (\text{Non-extenders})\bar{\setminus} aM$. Namely we obviously have

$$(3.1) \quad C_A^{II} \setminus aM = (\text{Picas})\bar{\setminus} aM \subset (\text{Non-extenders})\bar{\setminus} aM = C_A^I \setminus aM$$

since aM is closed and since a pica in $(M \setminus aM)$ is clearly a non-extender, because minimal joins are C^1 -smooth within of $(M \setminus aM)$ by theorem 2.1b. It remains to prove

$((\text{Picas})\bar{\setminus} aM) \supset ((\text{Non-extenders})\bar{\setminus} aM)$. The statement of the last inclusion is equivalent with its contraposition, i.e. the conclusion

$(q \in M \setminus ((\text{Pic} \bar{A}) \cup \partial M)) \implies (q \text{ is extender relative to } A) \quad (3.2)$

For the proof of (3.2) and for the proof of theorem 3.1b we proceed in the following three steps which we show separately at the end of the proof.

Step 1. We show that we have a continuous vector field.

$$(M \setminus (A \cup (\text{Pic} \bar{A}) \cup \partial M)) \ni q \mapsto \dot{c}_q$$

$\dot{c}_q :=$ (unique common initial vector of all minimal joins from q to A).

$|\dot{c}_q|_q = 1$, $|\cdot|_q$ the norm respective the Riemannian metric g at the point q . Note minimal joins from a point q to the closed set A exist by assertion 2.2 and theorem 1a.

Step 2. We show, if the function $d(A, \cdot)$ is differentiable at a point $q \in (M \setminus (A \cup (\text{Pic} \bar{A}) \cup \partial M))$ then the gradient $d(A, \cdot)$ at the point q equals $-\dot{c}_q$.

Step 3. Show lemma A.1': Let f be a real valued function defined on an open subset O of \mathbb{R}^n . Further let $v: O \rightarrow \mathbb{R}^n$ be a continuous vectorfield on O . Now if f is locally Lipschitz continuous and if it's gradient at those points where it exists equals the vector from the vector field v then f is a C^1 -smooth function on O and the gradient of f equals v on O .

Combining the statements of the three steps and using the Lipschitz continuity of the function $d(A, \cdot)$ we get that $d(A, \cdot)$ is C^1 -smooth and gradient $d(A, \cdot)$ equals $-\dot{c}_q$ for all q in the open sets $M \setminus (\partial M \cup A \cup (\text{Pic} \bar{A})) \supset M \setminus (C_A^I \cup A \cup \partial M)$. The last inclu-

sion follows from the conclusion in (3.1). Thus we have proved theorem 3.1b except the maximality claim there. Obviously

$M \setminus (\partial M \cup A \cup (\text{Pic} \bar{s}))$ is the maximal open subset of $M \setminus (\partial M \cup A)$ where $d(A, \cdot)$ is C^1 -smooth since $d(A, \cdot)$ clearly cannot be C^1 -smooth in a point $q \in (\text{Pic} \bar{s})$. Therefore we are finished with the proof of theorem 3.1a as well as with the proof of theorem

3.1b if we show (3.2). For this let $\tilde{q} \in M \setminus (\partial M \cup A \cup (\text{Pic} \bar{s}))$. We have to show that \tilde{q} is an extender. Take any minimal join

$\bar{c}: [0, d(A, \tilde{q})] \rightarrow M$ from A to \tilde{q} . Then within of $M \setminus (\partial M \cup A \cup (\text{Pic} \bar{s}))$ the path $\bar{c}(t)$ is an integral curve of the continuous vector field $q \mapsto -\dot{c}_q$. Namely for all $q = \bar{c}(t) \in M \setminus (\partial M \cup A \cup (\text{Pic} \bar{s}))$ we have $\dot{\bar{c}}(t) = \text{grad } d(A, \cdot) = -\dot{c}_q$.

We use the fact that \tilde{q} has a neighbourhood $U \subset M \setminus (\partial M \cup A \cup (\text{Pic} \bar{s}))$ say U homeomorphic to \mathbb{R}^n , whereon the vector field $q \mapsto -\dot{c}_q$ and hence the related differential equation are continuous.

By Peano's theorem we can extend the solution $\bar{c}(t)$ beyond \tilde{q} up to some point say $\bar{c}(d(A, \tilde{q}) + \delta)$, $\delta > 0$. Since $\dot{\bar{c}}(t) = \text{grad } d(A, \cdot)$ for all $\bar{c}(t)$, $t \in [0, d(A, \tilde{q})] \cup I$, $I := [d(A, \tilde{q}), d(A, \tilde{q}) + \delta]$, the path $\bar{c}: ([0, d(A, \tilde{q})] \cup I) \rightarrow M$ is really a distance realizing extension of the minimal join \bar{c} from A to \tilde{q} up to the point $\bar{c}(d(A, \tilde{q}) + \delta)$.

Proof of the claim in step 1: Let (q_n) , $n \in \mathbb{N}$ be any sequence of points converging against some point q_0 with $\{q_n | n \in \mathbb{N}_0\} \subset (M \setminus (A \cup (\text{Pic} \bar{s})) \cup \partial M)$. Take any sequence of normalized minimal joins $c_{q_n}(t)$ from q_n to A . We have to prove that

$\lim_{n \rightarrow \infty} \dot{c}_{q_n}(0) = \dot{c}_{q_0}$. Assume $\dot{c}_{q_n}(0)$ contains an equally denoted subsequence converging against say $v_{q_0} \neq \dot{c}_{q_0}$. Now we will derive a contradiction from the condition that q_0 belongs to $(M \setminus (A \cup (\text{Picas} \cup M)))$, i.e. \dot{c}_{q_0} is the unique common initial vector of all minimal joins from q_0 to A . Namely take some $\epsilon > 0$, such that relative to some chart (ϕ, U) for large enough numbers n the segments $c_{q_n}([0, \epsilon])$ are all contained in the neighbourhood U of q_0 , $\phi(U)$ a compact, convex body in $(\mathbb{R}^n, |\cdot|)$. In a bounded set any sequence of minimal paths contains a subsequence converging against some minimal path by assertion 2.1. Here the related subsequence is converging against some minimal join $c_{q_0}(t)$ from q_0 to A . Therefore we have an (equally denoted) subsequence of the normalized segments $c_{q_n}([0, \epsilon])$, $n \in \mathbb{N}$ with

$$(3.3) \quad \lim_{n \rightarrow \infty} \max_{t \in [0, \epsilon]} \{ |c_{q_n}(t) - c_{q_0}(t)| \} = 0,$$
 where $|\cdot|$ the norm related to the chart (ϕ, U) . Note $\dot{c}_{q_0}(0) = \dot{c}_{q_0}$! Now by theorem 2.1b the derivatives $\dot{c}_{q_n}(t)$, $n \in \mathbb{N}_0$, fulfill a Hölder 1/2 condition, i.e. $|\dot{c}_{q_n}(t) - \dot{c}_{q_n}(\bar{t})| \leq B |t - \bar{t}|^{1/2}$ for all $t, \bar{t} \in [0, \epsilon]$, B valid for all minimal joins contained in $\phi(U)$. Using this Hölder 1/2 continuity and the assumption $\lim_{n \rightarrow \infty} \dot{c}_{q_n}(0) \neq \dot{c}_{q_0} = \dot{c}_{q_0}(0)$, a routine estimation shows that there exist $\gamma, \delta \in]0, \epsilon[$ such that $|c_{q_n}(\gamma) - c_{q_0}(\gamma)| > \delta$ for all $n \in \mathbb{N}$. This contradicts (3.3).

Proof of the claim in step 2: We obviously have $|\text{grad } d(A, q)|_q \leq 1$.

Assume $\text{grad } d(A, \cdot) \neq \dot{c}_q$ at some point q . Denote the directional derivative of $d(A, \cdot)$ in direction of the vector $(-\dot{c}_q)$ by $(-\dot{c}_q)d(A, \cdot)$. Now we get a contradiction since

$$(-\dot{c}_q)d(A, \cdot) = \langle \text{grad } d(A, q), -\dot{c}_q \rangle_q < |\text{grad } d(A, q)|_q |-\dot{c}_q|_q \leq 1$$

and because $(-\dot{c}_q)(d(A, \cdot)) = -\lim_{t \rightarrow 0} \frac{d(A, c_q(t)) - d(A, q)}{t} = 1,$

$\langle \cdot, \cdot \rangle_q$ being the Riemannian metric at the point q .

Proof of the claim in step 3: Lemma A.1' is a special case of Lemma A.1, which we prove in the appendix.

Theorem 3.2: Let (M, d) be a space as assumed in theorem 2.1b. Then $M \setminus (C_A^{\text{III}} \cup A)$ is the maximal open set in $M \setminus A$, on which the function $d(A, \cdot)$ is C^1 -smooth.

Proof of theorem 3.2: Our proof of theorem 3.2 relies essentially on arguments used in the preceding proof of theorem 3.1. By assertion 2.2 and by theorem 2.1b any point $q \in (M \setminus A)$ has an everywhere C^1 -smooth minimal join $c_q(t)$ from q to the set A . Due to definition 3.4.III we can define a vector field $q \mapsto -\dot{c}_q$ on $M \setminus (C_A^{\text{III}} \cup A)$ in the same manner as in step 2 in the above proof of theorem 3.1. Using the assumption that (M, d) is locally C^1 -diffeomorphic to convex sets in \mathbb{R}^n , the same argument as in the above proof for the claim in step 1 shows that the vector field $q \mapsto \dot{c}_q$ is continuous on $M \setminus (C_A^{\text{III}} \cup A)$. Further the same argument used for the claim

in step 2 above shows that at a point $q \in ((M \setminus (A \cup C_A^{\text{III}})) \setminus \partial M)$ where $d(A, \cdot)$ is differentiable $\text{grad } d(A, \cdot)$ equals $-\dot{c}_q$. Combining these facts and the Lipschitz continuity of $d(A, \cdot)$ with the following lemma A.1 (which we prove in the appendix) yields immediately that $d(A, \cdot)$ is C^1 -smooth on $M \setminus (A \cup C_A^{\text{III}})$.

Lemma A.1: Let K be a locally convex body in \mathbb{R}^n .

Let $v: K \rightarrow \mathbb{R}^n$ be a continuous vectorfield and let $f: K \rightarrow \mathbb{R}$ be a locally Lipschitz continuous function. If gradient f at those points where it exists within $K \setminus \partial K$ equals the vector from the vector field v , then f is C^1 -smooth on K and $\text{grad } f$ equals v on K .

Now since $d(A, \cdot)$ can not be C^1 -smooth in a point where two minimal joins to A start with distinct tangents, we get that $M \setminus (C_A^{\text{III}} \cup A)$ is really the maximal open set in $M \setminus A$ where $d(A, \cdot)$ is C^1 -smooth.

It is obvious that we obtain by theorem 3.1 and by theorem 3.2 some kind of generalized Gauss lemma i.e: Outside the cut locus C_A a level surface of the distance function $d(A, \cdot)$ being a C^1 -smooth hypersurface is orthogonal to the minimal joins from A to points in this hypersurface.

Corollary 3.3 (Generalized Gauss Lemma): Let $A \subset M$ be closed.

Let r be any positive number, we define $S_r(A) := \{q \in M \mid d(A, q) = r\}$.

a) Let (M, d) be a space as given in the assumption of theorem 2.1 and

let O be an open set in (M, d) with $C_A^I \cap (O \setminus \partial M) = \emptyset$. Then for every $r > 0$ $(U \setminus \partial M) \cap S_r(A)$ is a C^1 -smooth hypersurface if $(U \setminus \partial M) \cap S_r(A) \neq \emptyset$. Further $(U \setminus \partial M) \cap S_r(A)$ is orthogonal to the minimal joins from A to points in $(U \setminus \partial M) \cap S_r(A)$.

b) Let (M, d) be a space as assumed in theorem 2.1b) and let O be an open set in (M, d) with $C_A^{III} \cap O = \emptyset$. Then every point $q \in O$ has a neighbourhood $U(q)$ in M such that the following holds:

For all $r > 0$ with $U(q) \cap S_r(A) \neq \emptyset$ exists a C^1 -smooth hypersurface H_r in the ambient manifold \tilde{M} and $U(q) \cap S_r(A) = U(q) \cap H_r$. Further H_r is orthogonal to the minimal ¹⁾ joins from A to points in $U(q) \cap S_r(A)$.

Proof: The proof of a) is a trivial application of theorem 3.1

For the proof of b) recall by theorem 3.2 $d(A, \cdot)$ is C^1 -smooth on all O .

We need now the property of M that every point $q \in M$ has a neighbourhood $\tilde{U}(q), \tilde{U}(q)$ being diffeomorphic to a convex set in \mathbb{R}^n . Clearly we can assume $\tilde{U}(q) \subset O$. Now applying Whitney's extension theorem [1] p. 120 it is not difficult to see that we can extend the function

1) We mean here minimal joins in the space (M, d) .

$d(A, \cdot): \tilde{U}(q) \rightarrow \mathbb{R}^1$) to a C^1 -smooth function defined on an open set \tilde{U}^* in \tilde{M} with $\tilde{U}^* \supset \tilde{U}(q)$. Using the implicit function theorem it is now easy to finish the proof of b).

Remark: Let from now on C_A be the cut locus in the sense of definition 3.4.I. Thus we have always $C_A = \hat{C}_A^I$ unless otherwise said.

1) $\tilde{U}(q)$ denotes the closure of $\tilde{U}(q)$, clearly $\tilde{U}(q)$ is diffeomorphic to a convex body in \mathbb{R}^n . One needs this convexity to verify the conditions of Whitney's extension theorem.

§ 3a GLUEING SEAM LEMMA

In the preceding paragraph we proved in theorem 3.1a that the picas are dense in the set of non-extendors. This result was a generalisation of a result which we had already proved in [76] lemma 2. Lemma 2 in [76] says: "For any complete unbordered Riemannian manifold M and any point $p \in M$ those points with at least two minimal joins to p are dense in the cut locus C_p of the point p ." Our proof of lemma 2 in [79] using the invariance of domain theorem relied on the continuity of the map $s: \{x/x \in T_p M, |x|=1\} \rightarrow [0, \infty]$, $s(x) := \sup\{\alpha \in \mathbb{R} / d(p, \exp_p(\alpha x)) = \alpha\}$, $[0, \infty]$ the one point compactification of the intervall $[0, \infty)$. That proof of lemma 2 was topologically motivated. Indeed the possibility for a topological interpretation of lemma 2 in [76] becomes very clear by the subsequent lemma 3a.1, the so called "glueing seam lemma".

Lemma 3a.1: Let M be a compact unbordered topological manifold, $f: B := \{x \in \mathbb{R}^n / |x| \leq 1\} \rightarrow M$ a continuous surjective map with the restriction of f being injective on $B \setminus \partial B$, $|\cdot|$ the Euclidean norm. Then the "proper identification points" of the map f i.e. the points $q \in f(\partial B)$ with $^1) \text{card } f^{-1}(q) \geq 2$ are dense in the "glueing seam" $f(\partial B)$.

We consider $f(\partial B)$ to be a "glueing seam", where the disc B via identification on it's boundary is glued to become the manifold M . In our topological interpretation

¹⁾ We denote by $\text{card } B$ the number of elements in a set B .

of lemma 2 in [7] the exponential map \exp_p corresponds to the above map f in lemma 3a.1' and the cut locus of p corresponds to $f(\partial B)$. Clearly if M is compact then lemma 2 is an immediate consequence of the mere topological lemma 3a.1'. We shall prove the following lemma 3a.1, which will give us also lemma 3a.1'.

Lemma 3a.1: Let M be a compact topological manifold, ∂M may be nonempty. Let $f: B := \{x \in \mathbb{R}^n / |x| \leq 1\} \rightarrow M$ be a continuous and surjective map and assume that the restriction of f is injective on $\overset{\circ}{B} := \{x \in \mathbb{R}^n / |x| < 1\}$. Then the set of points $D := \{x \in B / \text{card } f^{-1}(f(x)) \geq 2, f(x) \notin \partial M\}$ is dense in $S := \{x \in B / |x| = 1, f(x) \notin \partial M\}$.

Using the continuity of the map f the preceding lemma yields immediately the subsequent corollary 3a.1 which includes lemma 3a.1'.

Corollary 3a.1: The "proper identification points" i.e. the points $y \in M$ with $\text{card } f^{-1}(y) \geq 2$ are dense in the glueing seam i.e. in $f(S) \setminus \partial M$.

Proof of lemma 3a.1: The proof will be indirect. Assume there exists a point $x_0 \in S$ such that x_0 cannot be approximated by a sequence of points contained in D . Then we have a number $\delta > 0$ such that $H_\delta(x_0) := B_\delta(x_0) \cap B$ is homeomorphic to B and $D \cap H_\delta(x_0) = \emptyset$, $B_\delta(x_0) := \{x \in \mathbb{R}^n / |x - x_0| \leq \delta\}$. Thus the restriction of f is injective on $H_\delta(x_0)$. Therefore the restriction of the map f to the set $H_\delta(x_0)$ is a homeomorphism onto it's

image $f(H_\delta(x_0))$, (being a continuous univalent mapping on a compact set). Let $\varphi: U(f(x_0)) \rightarrow \mathbb{R}^n$ be a chart for a neighbourhood of $f(x_0)$. We identify $U(f(x_0))$ with the related set in the chartspace. There exists a natural number N_0 such that for any natural number $n > N_0$ the open ball $\overset{\circ}{B}_{\frac{1}{n}}(f(x_0)) := \{x \in \mathbb{R}^n \mid |x - f(x_0)| < \frac{1}{n}\}$ is contained in $U(f(x_0))$. Now for any natural number n larger than a certain number $N_0 \geq N_0'$ the open ball $B_{\frac{1}{n}}(f(x_0))$ contains points lying in the complement of $f(H_\delta(x_0))$ because x_0 has in $H_\delta(x_0)$ a neighbourhood homeomorphic to the closed n -dimensional Euclidean halfspace. Thus we can take a sequence of points (y_n) $n \geq N_0$ with y_n in $B_{\frac{1}{n}}(f(x_0))$ but not in $f(H_\delta(x_0))$. We define a sequence (x_n) by taking for every number $n \geq N_0$ a point $x_n \in f^{-1}(y_n)$. The sequence (x_n) has a cluster point \bar{x}_0 in B . By passing to an equally denoted subsequence if necessary we can assume that (x_n) converges against \bar{x}_0 . Clearly $\bar{x}_0 \neq x_0$ because $x_n \notin H_\delta(x_0)$ for all $n > N_0$. However due to the continuity of the map f we get $f(x_0) = \lim (y_n) = \lim f(x_n) = f(\lim (x_n)) = f(\bar{x}_0)$. Thus $\text{card } f^{-1}(f(x_0)) \geq 2$. Therefore $x_0 \in D$ contradicting the assumption that x_0 is not a cluster point of D .

§ 4 Lipschitz points and simulation of certain conjugate points

A well known classical result for the cut locus C_p relative to a point p in an unbordered complete C^∞ -smooth Riemannian manifold can be expressed in our terminology as follows: 'If a point $q \in C_p$ is not a pica relative to p , then q must be a conjugate point relative to p .'

This means the point q must be a singular value of the exponential map \exp_p . In our situation in bordered manifolds an exponential map is in general not available.

Nevertheless analysing the behaviour of geodesics which arrive in the neighbourhood of a point q in the cut locus, where q is not a pica, we find out how to simulate those conjugate points or one might say how to imitate those singular values of the exponential map. See in particular (4.1) and (4.2) below. Vice versa expressing that a point is neither such a conjugate point nor a pica we are led to the concept of a Lipschitz point. For this during the whole paragraph let (M,d) be a space of type $(\Delta)^1$, however assume \hat{M} to be a C^∞ -smooth manifold with C^∞ -smooth Riemannian metric g unless otherwise said and let A be a closed subset of (M,d) .

Definition 4.1: A point $q \in M \setminus (\partial M \cup A)$ is called Lipschitz point relative to some closed set A or Lipschitz point in short, if there exists a number R such that in some chart $|\dot{c}_q - \dot{c}_{\bar{q}}| \leq R d(q, \bar{q})$ with $\dot{c}_{\bar{q}}$ the tangent vector at \bar{q} of any normalized minimal join from \bar{q} to A , $|\cdot|$ being the norm related to the chart.

1) cf. theorem 2.1.

Definition 4.1': A point $q \in M \setminus (\partial M \cup A)$ which is not a Lipschitz point relative to A is called non-Lipschitz point relative to A or non-Lipschitz point in short.

Remark 4.1: The definition 4.1 excludes e.g. obviously that a Lipschitz point is a pica relative to A . However we do not require in this definition that initial vectors of minimal joins starting at points in the neighbourhood of q are uniquely determined by their starting point. E.g. the point q_0 in figure 3.2 and the point q_0 in figure 3.3 are Lipschitz points which contain picas in every neighbourhood. Clearly the point q_0 in both figures is an extender. That the point q_0 in those figures is a Lipschitz point can be seen directly from the main result in this paragraph, the simplified (and weakened) version of which may be given as follows.

Theorem 4.1: A point $q \in M \setminus (\partial M \cup A)$ is an extender relative to A iff q is a Lipschitz point relative to A .

Note theorem 4.1 need not to hold for a point $q \in \partial M \setminus A$. Namely applying the definition of Lipschitz point and extender literally also to boundary points the example in figure 3.4 shows, that the point q_2 on M there is an extender and at the same time a non-Lipschitz point. It follows from theorem 4.1: 'If there exists a sequence q_n with $\lim q_n = q_0$ such that with the notation used in definition 4.1,

$$(4.1) \quad \lim_{n \rightarrow \infty} \frac{d(q_0, q_n)}{|c_{q_0} - c_{q_n}|} = 0,$$

then q_0 is a nonextender.'

This last statement may perhaps help to explain how we come to the concept of Lipschitz point after analysing the situation in an unbordered manifold where say the point q_0 belongs to the cut locus C_p relative to some point p and where q_0 is a singular value of the exponential map \exp_p . The crucial part of our proof of theorem 4.1 may be seen in the proof of the following statement, which is a simplified version, of Lemma 4.1 below.

Lemma 4.1: Let $q \in M \setminus (\partial M \cup A)$ be an extender relative to A . Denote by $c_{\bar{q}}(t)$ any normalized minimal join from \bar{q} to A , then there exists $\tilde{\epsilon} > 0$ such that for each $\epsilon \in]0, \tilde{\epsilon}[$ exists $\delta(\epsilon) > 0$ with

$$(4.2) \quad \frac{d(q, \bar{q})}{d(c_q(\epsilon), c_{\bar{q}}(\epsilon))} \geq \delta(\epsilon)$$

for all $\bar{q} \in B_{\tilde{\epsilon}}(q) := \{q' \mid q' \in M, d(q, q') \leq \tilde{\epsilon}\}$.

Theorem 4.1 yields directly the following characterisation of cut loci.

Corollary and theorem 4.2: The cut locus C_A^I i.e. the cut locus relative to the set A in sense of definition 3.4.I is the closure of all points which are non-Lipschitz points relative to A .

In an unbordered C^∞ -smooth Riemannian manifold it cannot happen that a sequence of non-extenders relative to some point p is converging against an extender relative to p . Therefore we have by theorem 4.1 in an unbordered complete C^∞ -smooth Riemannian manifold the following characterisation of the cut locus C_p relative to p .

Corollary and theorem 4.3: In an unbordered C^∞ -smooth Riemannian manifold the cut locus relative to some point p is the set of all non-Lipschitz points relative to p .

Remark 4.2: In relation to the last corollary we wish to point out the following facts. In a bordered manifold it may well happen that a sequence q_n of non-Lipschitz points relative to some point p is converging against a Lipschitz point $q_0 \in M \setminus (\partial MU\{p})$ relative to p , see figure 3.2. If we take instead of a single point p some closed set A then it may even happen in an unbordered complete manifold that a sequence q_n of non-Lipschitz points relative to A converges against a Lipschitz point q_0 relative to A , see figure 3.3. Moreover the just described phenomenon may even happen if we take in an unbordered complete manifold for the set A a compact $C^{1,1}$ -smooth submanifold, see also § 5p. 12.7. Note in an unbordered complete Riemannian manifold any closed $C^{1,1}$ -smooth submanifold is automatically, a subset enjoying the so called unique footpoint property, see definition 5.1.

E. Kaufmann has shown in [37] that there exists a compact submanifold A of \mathbb{E}^2 with the following properties. Namely A being homeomorphic to S^1 enjoys the so called local unique footpoint property and the picas relative to A are dense in some open subset O of \mathbb{E}^2 . Since any extender is a Lipschitz point by theorem 4.1 and because every point in O has a minimal join to A and can therefore be approximated by extenders, Kaufmann's example shows, that we have in O a dense subset of Lipschitz points and also a dense subset of non-Lipschitz points relative to A . Moreover if we take for some small enough positive number ϵ a certain parallel surface $A_{\epsilon+}$ of the set A in Kaufmann's example, $A_{\epsilon+}$ being a connected component of the set $\{x \in \mathbb{E}^2 \mid d(A, x) = \epsilon\}$, then $A_{\epsilon+}$ is a $C^{1,1}$ -smooth submanifold of \mathbb{E}^2 , cf. theorem 5.1 and we have that picas and Lipschitz points relative to $A_{\epsilon+}$ agree in the open set O with those picas and Lipschitz points relative to A .

If we apply the definition of non-Lipschitz point and extender literally also to boundary points then q_0 in figure 3.4 is a non-Lipschitz point and extender. - Note in general we can say if the point $q_0 \in \partial M$ is a branching point relative to A , then q_0 cannot have a neighbourhood $U(q_0)$ in M , consisting of Lipschitz points relative to A with some common Lipschitz constant R , valid for all points in $U(q_0)$. This holds due to the uniqueness properties of solution curves of Lipschitz continuous vector fields. It may however happen

that all single points in some neighbourhood $U(q_0)$ of a branching point q_0 are Lipschitz points. In order to demonstrate this take the following example, where M is a bordered submanifold of the Euclidean plane consisting of all points which are located below and on the graph of the function $y = x^2$. Now let $p \in M$ be a point with Euclidean coordinates say $p = (-10, 0)$. Then it is easily seen that all points in $M \setminus (\{p\} \cup \partial M)$ are Lipschitz points relative to p . Even the point $q_0 := (0, 0) \in \partial M$ being a branching point relative to p is a Lipschitz point relative to p if one applies definition 4.1 literally to q_0 . However, a simple calculation shows: For every $\epsilon > 0$ and for every natural number n exist two numbers $\delta_n, \bar{\delta}_n, 0 < \delta_n < \bar{\delta}_n < \epsilon^2$ such that $|\dot{c}_{q_n} - \dot{\bar{c}}_{q_n}| \geq n |q_n - \bar{q}_n|$ for the points $q_n := (\epsilon, \epsilon^2 - \delta_n), \bar{q}_n := (\epsilon, \epsilon^2 - \bar{\delta}_n)$ $\dot{c}_{q_n}, \dot{\bar{c}}_{q_n}$ the initial vectors of the normalized minimal joins from q_n, \bar{q}_n to p .

We now come to a series of background results of theorem 4.1. The following definition, roughly spoken quantitative version of definition 3.1 will turn out to be comfortable and useful in future considerations.

Definition 4.2: A point $q \in M \setminus \partial M$ is called ϵ -extender relative to some closed set A , if there exists a non trivial minimal join from A to q which can be extended minimally by length ϵ beyond q with the extension contained in $M \setminus \partial M$.

Crucial for most results in this paragraph is the subsequent lemma 4.1 which includes lemma 4.1'. Despite its technical appearance this lemma 4.1 has a proof which contains some basic geometric ideas of this paragraph.

Lemma 4.1: Let $q_0 \in M \setminus \partial M$ and let $r > 0$ be such that $\exp_{q_0} : K_r(0) \rightarrow \exp_{q_0}(K_r(0)) \subset M \setminus \partial M$ is a diffeomorphism. 1) Then there exists $\varepsilon_0 > 0$ and for every $\varepsilon \in]0, \varepsilon_0[$ a number $\beta > 0$ such that the following holds:

If $A \subset M$ is closed and $q \in B_r(q_0)$ an ε -extender with respect to A , then

$$\frac{d(q, \bar{q})}{d(c_q(\varepsilon), c_{\bar{q}}(\varepsilon))} \geq \beta \quad \text{for all } \bar{q} \in B_{\frac{\varepsilon}{2}}(q),$$

if $B_{\frac{3\varepsilon}{2}}(q) \cap A = \emptyset$.

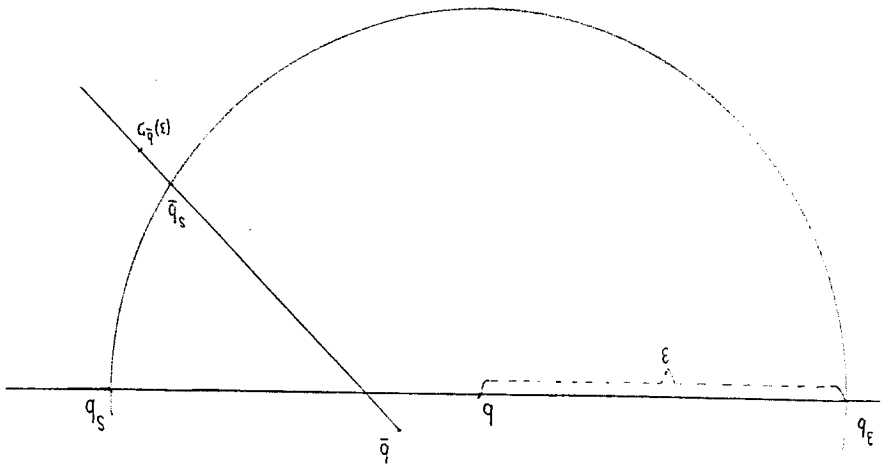


Figure 4.1

1) Note $K_r(0) := \{v \in T_{q_0} M / |v| \leq r\}$. Thus we have here

$$B_r(q_0) := \{q' \in M / d(q_0, q') \leq r\} = \exp_{q_0}(K_r(0)) \subset M \setminus \partial M.$$

Proof of Lemma 4.1: We shall give detailed conditions for the number ε_0 later on. Let us for the moment assume that the point $q \in B_r(q_0)$ is an ε -extender relative to A . Let $\varepsilon > 0$ be small enough such that $B_{\frac{3\varepsilon}{2}}(q) \cap A = \emptyset$ and that $B_\varepsilon(q)$ is contained in a domain of Riemannian normal coordinates around q . 1) Pick a geodesic sphere $S_\varepsilon(q) := \{y / d(y,q) = \varepsilon\}$, see the above figure 4.1 which we use to illustrate our considerations. Now let \bar{q} be any point with $d(\bar{q},q) \leq \frac{\varepsilon}{2}$. The minimal joins $c_q(t)$, $c_{\bar{q}}(t)$ (going from q, \bar{q} to A) will meet $S_\varepsilon(q)$ the first time at points q_s, \bar{q}_s respectively, note $c_q(\varepsilon) = q_s$. The proof is performed in several steps.

1. Step: It is sufficient to estimate $\frac{d(\bar{q},q)}{d(\bar{q}_s, q_s)}$

from below because

$$(4.3) \quad \frac{d(\bar{q},q)}{d(c_{\bar{q}}(\varepsilon), c_q(\varepsilon))} \geq \min \left\{ \frac{1}{2}, \frac{1}{2} \frac{d(\bar{q},q)}{d(\bar{q}_s, q_s)} \right\}$$

Proof of (4.3):

$$\text{Assume (4.4)} \quad \frac{d(\bar{q},q)}{d(c_{\bar{q}}(\varepsilon), c_q(\varepsilon))} < \frac{1}{2}.$$

Then we get

1) Note $B_\varepsilon(q) \subset M \setminus \partial M$.

$$\begin{aligned}
 (4.5) \quad d(\bar{q}_s, q_s) &\geq d(q_s, c_{\bar{q}}(\varepsilon)) - d(c_{\bar{q}}(\varepsilon), \bar{q}_s) \\
 &= d(c_{\bar{q}}(\varepsilon), c_q(\varepsilon)) - (\varepsilon - d(\bar{q}_s, \bar{q})) \\
 &= d(c_{\bar{q}}(\varepsilon), c_q(\varepsilon)) - (d(\bar{q}_s, q) - d(\bar{q}_s, \bar{q})) \\
 (4.6) \quad &\geq d(c_{\bar{q}}(\varepsilon), c_q(\varepsilon)) - d(\bar{q}, q) \\
 (4.7) \quad &> \frac{1}{2} d(c_{\bar{q}}(\varepsilon), c_q(\varepsilon)) \quad .
 \end{aligned}$$

Here (4.5) and (4.6) are derived using the triangle inequality and (4.7) is got using (4.4). Now (4.7) yields

$$\frac{d(\bar{q}, q)}{d(\bar{q}_s, q_s)} \leq \frac{d(\bar{q}, q)}{\frac{1}{2} d(c_{\bar{q}}(\varepsilon), c_q(\varepsilon))} \quad .$$

This proves (4.3).

2. Step: Localisation.

The fact that

(4.8) the path $c_q(t)$ can be extended minimally backward up to the point $q_\varepsilon := c_q(-\varepsilon)$

yields

$$\begin{aligned}
 (4.9) \quad d(q_s, q_\varepsilon) + d(\bar{q}_s, \bar{q}) &\leq d(q_s, \bar{q}) + d(\bar{q}_s, q_\varepsilon) \quad , \\
 (4.9') \quad 2\varepsilon &\leq d(q_s, \bar{q}) - d(\bar{q}_s, \bar{q}) + d(\bar{q}_s, q_\varepsilon) \quad .
 \end{aligned}$$

Proof of (4.9) and (4.9'): Now (4.8) implies (4.8') below. Therefore using the triangle inequality we get

$$\begin{aligned}
 (4.10) \quad d(A, q_s) + d(q_s, q_\varepsilon) &\stackrel{(4.8')}{=} d(A, q_\varepsilon) \leq d(A, \bar{q}_s) + d(\bar{q}_s, q_\varepsilon) \quad , \\
 (4.10') \quad d(A, \bar{q}_s) + d(\bar{q}_s, \bar{q}) &= d(A, \bar{q}) \leq d(A, q_s) + d(q_s, \bar{q}) \quad .
 \end{aligned}$$

Summing up (4.10) and (4.10') we easily get (4.9).
 Inequality (4.9') follows immediately from (4.9) because $d(q_S, q_\varepsilon) = 2\varepsilon$.

In order to present the crucial ideas of the proof in a transparent way we complete now first the proof of lemma 4.1 in the Euclidean case.

Proof of Lemma 4.1 in the Euclidean case:

If the ambient manifold \hat{M} of M is a Euclidean space then (4.9) yields

$$(4.11) \quad \frac{d(\bar{q}, q)}{d(\bar{q}_S, q_S)} \geq \frac{\sqrt{1 + \frac{2\varepsilon}{\varepsilon - \rho}} - 1}{\frac{4\varepsilon}{\varepsilon - \rho}} \quad \text{for all } \bar{q} \in B_\rho(q),$$

$$0 < \rho < \varepsilon,$$

$$(4.11') \quad > \frac{1}{8} \quad \text{if } \rho \in]0, \frac{\varepsilon}{2}] .$$

We shall prove (4.11) later. Now we prove that inequality (4.11') follows from (4.11). Let

$$h(z) := \frac{\sqrt{1 + 2z} - 1}{4z}, \quad z(\rho) := \frac{\varepsilon}{\varepsilon - \rho} .$$

Then we have $z(0) = 1$, $z(\frac{\varepsilon}{2}) = 2$. Thus $h(z(0)) = h(1) > \frac{1}{8}$, $h(z(\frac{\varepsilon}{2})) = h(z) > \frac{1}{8}$. A straight forward computation shows that

$$\frac{d z(\rho)}{d \rho} > 0, \quad \text{if } 0 \leq \rho \leq \frac{\varepsilon}{2}, \quad \text{with } \varepsilon > 0, \quad \text{and that}$$

$$\frac{d h(z)}{d z} < 0, \quad \text{if } z > 0 .$$

Hence $z(\rho)$ is monoton increasing and $h(z)$ is monoton de-

creasing if $0 \leq \rho \leq \frac{\varepsilon}{2}$ and $z > 0$. Therefore

$$\{z(\rho) / 0 \leq \rho < \frac{\varepsilon}{2}\} = [1, 2[\text{ and } \{h(z) / z \in [1, 2[\subset]\frac{1}{8}, \frac{\sqrt{5}-1}{8}].$$

This proves (4.11').

Proof of 4.11: We have $d(x, y) = \|x - y\|$, $\|\cdot\|$ the Euclidean norm. Therefore and because $d(q_s, \bar{q}_s) = 2\varepsilon$ inequality (4.9) yields

$$(4.12) \quad 2\varepsilon \leq \|q_s - \bar{q}\| - \|\bar{q}_s - \bar{q}\| + \|\bar{q}_s - q_s\|.$$

We go on to estimate the right hand side of (4.12). Applying the Taylor formula to the function

$$f(x) := \|q_s - x\| - \|\bar{q}_s - x\|$$

we get

$$f(\bar{q}) = \|q_s - \bar{q}\| - \|\bar{q}_s - \bar{q}\| = f(q) + D_q f(\bar{q} - q) + \frac{1}{2} D_p^2 f(\bar{q} - q)^2,$$

with p a point in $\{q + t(\bar{q} - q) / t \in [0, 1]\}$.

Now we have

$$\begin{aligned} D_p f(v) &= - \left\langle \frac{q_s - p}{\|q_s - p\|}, v \right\rangle + \left\langle \frac{\bar{q}_s - p}{\|\bar{q}_s - p\|}, v \right\rangle, \\ D_p^2 f(v)^2 &= - \frac{1}{\|q_s - p\|^3} \left\langle q_s - p, v \right\rangle^2 + \frac{1}{\|\bar{q}_s - p\|^3} \left\langle \bar{q}_s - p, v \right\rangle^2 \\ &\quad - \frac{1}{\|q_s - p\|} \left\langle v, v \right\rangle + \frac{1}{\|\bar{q}_s - p\|} \left\langle v, v \right\rangle \\ &\leq \|v\|^2 \left[\frac{1}{\|q_s - p\|} + \frac{1}{\|\bar{q}_s - p\|} \left\| \left\langle \frac{\bar{q}_s - p}{\|\bar{q}_s - p\|}, \frac{v}{\|v\|} \right\rangle \right|^2 \right], \end{aligned}$$

$\langle \cdot, \cdot \rangle$ the Euclidean scalar product. This yields

$$\begin{aligned}
 \|q_s - \bar{q}\| - \|\bar{q}_s - \bar{q}\| &\leq 0 + \frac{1}{\epsilon} \left\langle (\bar{q}_s - q) - (q_s - q), \bar{q} - q \right\rangle \\
 &\quad + \|\bar{q} - q\|^2 \left\{ \frac{1}{\epsilon - \rho} + \frac{1}{\epsilon - \rho} \right\} \\
 &\leq \frac{1}{\epsilon} \left\langle \bar{q}_s - q_s, \bar{q} - q \right\rangle + \|\bar{q} - q\|^2 \frac{2}{\epsilon - \rho} \\
 (4.13) \quad &\leq \frac{1}{\epsilon} \|\bar{q}_s - q_s\| \|\bar{q} - q\| + \frac{2}{\epsilon - \rho} \|\bar{q} - q\|^2
 \end{aligned}$$

Now we use Thale's theorem to estimate the third term in the right hand side of (4.12). Namely by Thale's theorem the segments joining q_s with \bar{q}_s and q_ϵ with \bar{q}_s meet at \bar{q}_s with a right angle. Therefore

$$\begin{aligned}
 (4.14) \quad \|\bar{q}_s - q_\epsilon\| &= \sqrt{(2\epsilon)^2 - \|\bar{q}_s - q_s\|^2} \\
 (4.14') \quad &\leq 2\epsilon \left(1 - \frac{1}{2} \frac{\|\bar{q}_s - q_s\|^2}{(2\epsilon)^2} \right) \\
 (4.14) \quad &\leq 2\epsilon - \frac{\|\bar{q}_s - q_s\|^2}{4\epsilon}
 \end{aligned}$$

Here (4.14') follows from (4.14) by Bernoulli's inequality. We use (4.14) and (4.13) to estimate the right hand side of (4.12) and get

$$\begin{aligned}
 2\epsilon &\leq \frac{1}{\epsilon} \|\bar{q}_s - q_s\| \|\bar{q} - q\| + \frac{2}{\epsilon - \rho} \|\bar{q} - q\|^2 + 2\epsilon - \frac{\|\bar{q}_s - q_s\|^2}{4\epsilon} \\
 1 &\leq 4 \frac{\|\bar{q} - q\|}{\|\bar{q}_s - q_s\|} + 8 \frac{\epsilon}{\epsilon - \rho} \frac{\|\bar{q} - q\|^2}{\|\bar{q}_s - q_s\|^2} \\
 \frac{1}{4 \left(1 + \frac{2\epsilon}{\epsilon - \rho} \frac{\|\bar{q} - q\|}{\|\bar{q}_s - q_s\|} \right)} &\leq \frac{\|\bar{q} - q\|}{\|\bar{q}_s - q_s\|} \tag{4.15}
 \end{aligned}$$

Now a simple computation shows that

$$\frac{1}{4(1 + Ay)} > \frac{\sqrt{1 + A} - 1}{2A}$$

if $0 < y < \frac{\sqrt{1 + A} - 1}{2A}$, $A > 0$.

Therefore (4.15) yields

$$\frac{\|\bar{q} - q\|}{\|\bar{q}_s - q_s\|} \geq \frac{\sqrt{1 + \frac{2\varepsilon}{\varepsilon - \rho}} - 1}{\frac{4\varepsilon}{\varepsilon - \rho}}$$

This proves (4.11) and completes the proof of the Euclidean case.

We proceed now with the proof of the general case of Lemma 4.1.

3. Step: Estimation of $d(q_s, \bar{q}) - d(\bar{q}_s, \bar{q})$ with the Taylor formula.

We shall give detailed conditions for ε_0 later. Let $\varepsilon_1 > 0$ be such that for all $q \in B_r(q_0)$ \exp_q is a diffeomorphism from the closed ε_1 ball in $T_q M$ onto its image contained in $M \setminus \partial M$. We use geodesic normal coordinates with center q and we use the identification with \mathbb{R}^n via these coordinates. Therefore $\|\cdot\|, \langle \cdot, \cdot \rangle$ are now the Euclidean norm and the Euclidean scalar product relative to these coordinates.

Let

$$f(x) := d(q_s, x) - d(\bar{q}_s, x).$$

Then we have for $\varepsilon \in]0, \varepsilon_1[$

$$f(q) = 0$$

and

$$D_q f(v) = \left\langle \frac{q - q_s}{\|q - q_s\|}, v \right\rangle - \left\langle \frac{q - \bar{q}_s}{\|q - \bar{q}_s\|}, v \right\rangle = \frac{1}{\epsilon} \left\langle \bar{q}_s - q_s, v \right\rangle.$$

Using the Taylor formula there exists a point

$p \in \{ y \in B_\epsilon(q) / y = q + t(\bar{q} - q), 0 \leq t \leq 1 \}$ such that

$$d(q_s, \bar{q}) - d(\bar{q}_s, \bar{q}) \leq \frac{1}{\epsilon} \left\langle \bar{q}_s - q_s, \bar{q} - q \right\rangle + \frac{1}{2} D_p^2 f(\bar{q} - q, \bar{q} - q)$$

$$(4.16) \quad \leq \frac{1}{\epsilon} \|\bar{q}_s - q_s\| \cdot \|\bar{q} - q\| + P \|\bar{q} - q\|^2,$$

if $\bar{q} \in B_{\frac{\epsilon}{2}}(q)$ and if

$$(4.17) \quad P \geq \sup \{ \|D_{(x,y)}^2 d(\cdot, \cdot)\| / d(x,y) \geq \frac{\epsilon}{2}, x, y \in B_\epsilon(q), q \in B_r(q_0) \}.$$

It remains to justify the existence of the bound P in

(4.17). For this let $(X_1(\tilde{q}), \dots, X_n(\tilde{q}))$, $\tilde{q} \in B_r(q_0)$ be an orthonormal moving frame on $B_r(q_0)$ ¹⁾, see e.g. [66]

p. 7.17. Let $K_\epsilon(0)$ be a closed Euclidean ball of radius $\epsilon \in]0, \epsilon_1]$ and let $y = (y_1, \dots, y_n)$, $x = (x_1, \dots, x_n)$ be vectors with their Euclidean coordinates. Assume $x, y \in K_\epsilon(0)$.

Define a map

$$\phi(q, w) := \phi(q, (x, y)) := d(\exp_q(\sum_{i=1}^n x_i X_i(q)), \exp_q(\sum_{i=1}^n y_i X_i(q))),$$

$$w = (x, y),$$

then $\phi(q, w)$ is obviously a smooth map on

1) For instance for $i \in \{1, \dots, n\}$ and $\tilde{q} \in B_r(q_0)$ define $X_i(\tilde{q})$ to be the parallel translate along the unique minimal geodesic from q_0 to \tilde{q} .

$B_r(q_0) \times (K_\varepsilon(O) \times K_\varepsilon(O) \setminus A_\varepsilon)$ if say

$$A_\varepsilon := \{ (x, y) \in K_\varepsilon(O) \times K_\varepsilon(O) / \|x - y\| < \frac{\varepsilon}{10} \}.$$

It is easily seen that

$$\frac{\partial^2}{\partial w^2} \phi(q, (x, y)) = D_{(x, y)}^2 d(\dots) \text{ with } d(\dots) : B_\varepsilon(q) \times B_\varepsilon(q) \rightarrow \mathbb{R},$$

$B_\varepsilon(q)$ being here the representation of $\{q' \in M / d(q', q) \leq \varepsilon\}$ in Riemannian normal coordinates with center q . Therefore we can define

$$P := \max \left\{ \left\| \frac{\partial^2}{\partial w^2} \phi(q, w) \right\| / (q, w) \in B_r(q_0) \times (K_\varepsilon(O) \times K_\varepsilon(O) \setminus A_\varepsilon) \right\}$$

and P fulfills inequality (4.17).

4. Step: Estimation of $d(\bar{q}_s, q_\varepsilon)$.

Let ε_1 be as above (in the third step). We claim that for $\varepsilon \in]0, \varepsilon_1[$ we have (respective geodesic normal coordinates with center q) the subsequent inequalities

$$(4.18) \quad d(\bar{q}_s, q_\varepsilon) \leq \| \bar{q}_s - q_s \| + \frac{1}{2} Q \varepsilon \| \bar{q}_s - q_s \|^2$$

$$(4.18') \quad \leq 2\varepsilon - \frac{\| \bar{q}_s - q_s \|^2}{4\varepsilon} + \frac{1}{2} Q \varepsilon \| \bar{q}_s - q_s \|^2$$

with Q being a positive constant depending on ε_1 .

Inequality (4.18') follows with Thale's theorem and

Bernoulli's inequality from (4.18), see (4.14'). We

prove now (4.18). For a proof different from the following

one see Lemma A.2 in the appendix.

Proof of (4.18):

Let $\alpha(q_\epsilon, \bar{q}_s) = \alpha$, see figure 4.2.

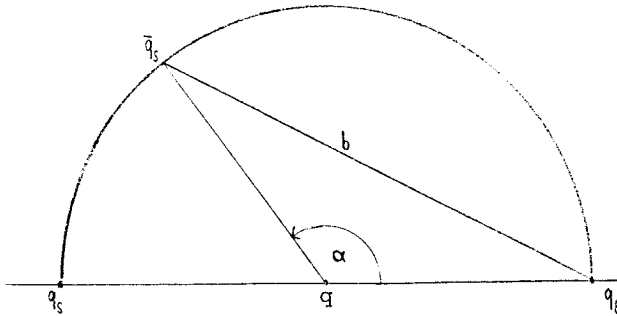


Figure 4.2

The Euclidean segment b from q_ϵ to \bar{q}_s is in the Euclidean plane $E(q, \bar{q}_s, q_\epsilon)$ ¹⁾ in polar coordinates described by

$$(4.19) \quad r(\phi) = \frac{a}{\cos(\frac{\alpha}{2} - \phi)} \quad , \quad 0 \leq \phi \leq \alpha \quad \text{with}$$

$$(4.19') \quad a = \epsilon \cos \frac{\alpha}{2} \quad , \quad \text{see figure 4.3.}$$

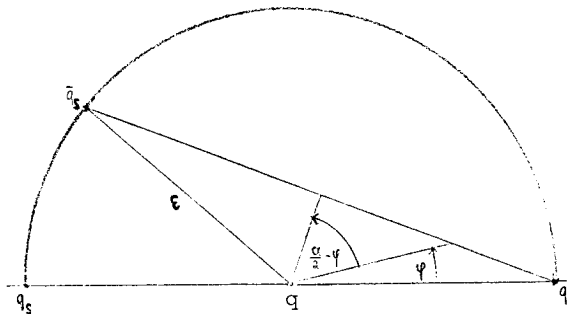


Figure 4.3

1) Note $E(q, \bar{q}_s, q_\epsilon)$ is the two-dimensional plane containing the three points q, \bar{q}_s, q_ϵ .

We have for the length L of $\exp_q^{-1}(b)$ ¹⁾

$$(4.20) \quad d(\bar{q}_s, q_\epsilon) \leq L = \int_0^\alpha \sqrt{\dot{r}^2 + g_{22}(r, \phi)} \, d\phi \quad .$$

with $\dot{r} = \frac{dr}{d\phi}$ and $g_{22}(r, \phi)$ related to the metric in the surface $\exp_q(E(q, \bar{q}_s, q_s) \cap B_\epsilon(q))$. It is well known that

$$\sqrt{g_{22}(r, \phi)} = r - \frac{\tilde{K}}{6} r^3 + \dots$$

\tilde{K} the curvature of the surface $\exp_q(E(q, \bar{q}_s, q_s))$ at the point q , cf. Lemma A₂ in the appendix.

Therefore there exists a positive number Q is depending on ϵ_1 such that

$$(4.21) \quad |g_{22}| \leq r^2 + 2 Qr^4$$

holds on $\exp_q(E(q, \bar{q}_s, q_s) \cap B_{\epsilon_1}(q))$ if $r \in]0, \epsilon_1]$;

here (4.21) with that number Q is valid for arbitrary points $q \in B_r(q_0)$ with \bar{q}_s, q_s being arbitrary points in $S_\epsilon(q) := \{ \tilde{q} \in M / d(\tilde{q}, q) = \epsilon \}$, cf.: proof of inequality (*) in Lemma A.2 in the appendix.

Let $\dot{r} := \frac{dr}{d\phi}$. Then (4.20) and (4.21) yield

$$d(\bar{q}_s, q_\epsilon) - \|\bar{q}_s - q_\epsilon\| \leq \int_0^\alpha (\sqrt{\dot{r}^2 + r^2 + 2Qr^4} - \sqrt{\dot{r}^2 + r^2}) \, d\phi =: A.$$

Using Bernoulli's inequality we get

$$A \leq \int_0^\alpha \frac{2Qr^4}{2\sqrt{\dot{r}^2 + r^2}} \, d\phi =: B \quad .$$

1) We mean here the length of the projection of b in the Riemannian manifold M .

Inserting (4.19) and using trigonometric routine computations we get

$$\begin{aligned}
 B &\leq Q \int_0^\alpha \frac{a^4}{\cos^4\left(\frac{\alpha}{2} - \phi\right)} \frac{1}{\sqrt{\left(\frac{-a}{\cos^2\left(\frac{\alpha}{2} - \phi\right)} \sin\left(\frac{\alpha}{2} - \phi\right)\right)^2 + \frac{a^2}{\cos^2\left(\frac{\alpha}{2} - \phi\right)}} d\phi \\
 &= Q \int_0^\alpha \frac{a^4}{\cos^4\left(\frac{\alpha}{2} - \phi\right)} \frac{1}{a \sqrt{\frac{1}{\cos^4\left(\frac{\alpha}{2} - \phi\right)}}} d\phi \\
 &= Q \int_0^\alpha \frac{a^3}{\cos^2\left(\frac{\alpha}{2} - \phi\right)} d\phi \\
 &= Q a^3 \left[-\tan\left(\frac{\alpha}{2} - \phi\right) \right]_0^\alpha \\
 &= Q a^3 2 \tan \frac{\alpha}{2} =: C .
 \end{aligned}$$

Inserting (4.19') we get from the last equation

$$\begin{aligned}
 C &= 2 Q \epsilon^3 \cos^3\left(\frac{\alpha}{2}\right) \frac{\sin \frac{\alpha}{2}}{\cos \frac{\alpha}{2}} \\
 &= 2 Q \epsilon^3 \cos^2\left(\frac{\alpha}{2}\right) \sin \frac{\alpha}{2} \\
 &= 2 Q \epsilon^3 \sin\left(\frac{\alpha}{2}\right) \frac{1}{2} (1 + \cos \alpha) \\
 &\leq \frac{1}{2} Q \epsilon 2 \epsilon^2 (1 + \cos \alpha) \\
 &\leq \frac{1}{2} Q \epsilon \|\bar{q}_s - q_s\|^2
 \end{aligned}$$

This proves 4.18.

5. Step: End of the proof of lemma 4.1.

Inserting the estimates from the 3rd step and 4th step i.e. precisely inserting (4.16) and (4.18') into (4.9') we get ¹⁾

$$(4.22) \quad 2\epsilon \leq \frac{1}{\epsilon} \|\bar{q}_s - q_s\| \cdot \|\bar{q} - q\| + P \|\bar{q} - q\|^2 + 2\epsilon - \frac{\|\bar{q}_s - q_s\|^2}{4\epsilon} + \frac{1}{2} Q d \|\bar{q}_s - q_s\|^2 .$$

This yields

$$1 - 2 Q \epsilon^2 \leq 4 \frac{\|\bar{q} - q\|}{\|\bar{q}_s - q_s\|} + 4 \epsilon P \frac{\|\bar{q} - q\|^2}{\|\bar{q}_s - q_s\|^2} .$$

Thus

$$\frac{\|\bar{q} - q\|}{\|\bar{q}_s - q_s\|} \geq \frac{1 - 2 Q \epsilon^2}{4 \left(1 + \epsilon P \frac{\|\bar{q} - q\|}{\|\bar{q}_s - q_s\|} \right)} .$$

Choose $\epsilon_0 \in]0, \epsilon_1]$ such that $2 Q \epsilon_0^2 < \frac{1}{2}$. Then

$$(4.23) \quad \frac{\|\bar{q} - q\|}{\|\bar{q}_s - q_s\|} \geq \frac{1}{8 \left(1 + \epsilon P \frac{\|\bar{q} - q\|}{\|\bar{q}_s - q_s\|} \right)}$$

for $\epsilon \in]0, \epsilon_0]$. Now a routine computation shows that if

$$0 < x < \frac{\sqrt{1 + \frac{A}{2}} - 1}{2 A}$$

1) Note here $\epsilon \in]0, \epsilon_1]$, q is an ϵ -extender and (4.22) holds if $d(q, \bar{q}) \leq \frac{\epsilon}{2}$.

then

$$\frac{1}{8(1+Ax)} > x .$$

This implies together with (4.23) that

$$\frac{\|\bar{q} - q\|}{\|\bar{q}_s - q_s\|} \geq \frac{\sqrt{1 + \frac{\epsilon}{2} P} - 1}{2 \epsilon P}$$

if $\epsilon \in]0, \epsilon_0]$.

By Lemma A4 in the appendix there exist two positive numbers G,F such that for every $q \in B_r(q_0)$ and for all $q_1, q_2 \in B_{\epsilon_0}(q)$

$$G d(q_1, q_2) \leq \|q_1 - q_2\| \leq F d(q_1, q_2)$$

with $\| \cdot \|$ the norm related to the normal coordinates with center q . Therefore

$$\frac{d(\bar{q}, q)}{d(\bar{q}_s, q_s)} \geq \frac{G \|\bar{q} - q\|}{F \|\bar{q}_s - q_s\|} \geq \frac{G}{F} \frac{\sqrt{1 + 2\epsilon P} - 1}{8 \epsilon P}$$

This yields together with (4.3) the following result:

If $q \in B_r(q_0)$ is an ϵ -extender with $\epsilon \in]0, \epsilon_0]$ then

$$\frac{d(q, \bar{q})}{d(c_q(\epsilon), c_{\bar{q}}(\epsilon))} \geq \min \left\{ \frac{1}{2}, \frac{G}{2F} \frac{\sqrt{1 + 2\epsilon P} - 1}{8 \epsilon P} \right\}$$

for all $\bar{q} \in B_{\frac{\epsilon}{2}}(q)$.

This completes the proof of Lemma 4.1.

Remark 4. In this paper we will make use of some identifications in order to simplify the notation. Namely if we have a chart (X,U) $U \subset M$, we will often identify a subset $B \subset U$ with its image $X(B)$ in \mathbb{R}^n ; $\| \cdot \|$ will always refer to the Euclidean norm ¹⁾ induced on B by the Euclidean structure on $X(B) \subset \mathbb{R}^n$.

1) Unless otherwise said.

We proceed now to prove several results which will finally lead to a somewhat globalized and sharpened version of theorem 4.1. For this we give next a proposition which does not contain a proper proof but explains concepts to be used in subsequent proofs.

Proposition 4.1 Let the Riemannian metric g be $C^{1,1}$ on \hat{M} , i.e. g has locally Lipschitz continuous first derivatives. Assume there is a chart (X, U) with $X(U) \supset C \supset X(B_{\text{or}}(q_0))$, $X^{-1}(C) \subset M$, C a compact convex set in \mathbb{R}^n and $\alpha > 1$. It is easily seen that there exist numbers f, F such that for all $q, \bar{q} \in B_r(q_0)$,

$$f \cdot d(q, \bar{q}) \leq |q - \bar{q}| \leq F \cdot d(q, \bar{q}),$$

$|\cdot|$ being the norm related to the chart (X, U) , see [77] p.29.

We describe now a Lipschitz continuous first order (system) differential equation for the geodesics in $B_r(q_0)$. This differential equation is used in lemma 4.2 below. For this denote the differential equation for the normalised geodesics respective the given coordinates (X, U) by $\ddot{x} = \Gamma(x, \dot{x})$. This is a second order (system) differential equation. Introducing new variables $x = x_1, \dot{x} = x_2$ we get a first order (system) differential equation

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} x_2 \\ \Gamma(x_1, x_2) \end{pmatrix} =: \bar{g} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

Now using the local Lipschitz continuity of the Christoffel symbols and the compactness of C it is easy to prove that there exists a number $L > 0$ with

$$\left| \bar{g} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} - \bar{g} \begin{pmatrix} x_1+h_1 \\ x_2+h_2 \end{pmatrix} \right| \leq L \left| \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} \right| \quad \text{for say all } (x_1+h_1) \in B_r(q_0),$$

$$\text{where } \left| \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} \right| := |\dot{x}_1| + |\dot{x}_2| \quad \text{and} \quad \left| \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \right| := |x_1| + |x_2|.$$

Note we use here also that we have $|x_2| := |\dot{x}(t)|$ bounded for all geodesic pieces contained in $B_r(q)$ since $\langle \dot{x}, (g_{x,j}(x))\dot{x} \rangle = 1$ and because the Riemannian metric $(g_{x,j}(x))$ is varying continuously with the footpoint x in the compact set $B_r(q)$, $\langle \cdot, \cdot \rangle$ being the Euclidean metric associated with the chart (X, U) .

Lemma 4.2: Let the Riemannian metric g have locally Lipschitz continuous derivatives. Let (X,U) be a chart for M , $B \subset \bar{B} \subset U$ with $X(\bar{B})$ compact and convex. Let $L > 0$ be a Lipschitz constant on B for the geodesic differential equation with respect to (X,U) . Let $\epsilon > 0$ and $c, \bar{c} : [0, \epsilon] \rightarrow B$ be two geodesics, and assume

$$2\epsilon Le^{\epsilon L} \leq 1 .$$

Then for any $\beta > 0$

$$(4.24) \quad |c(\epsilon) - \bar{c}(\epsilon)| \leq \beta |c(0) - \bar{c}(0)|$$

implies

$$(4.25) \quad |\dot{c}(0) - \dot{\bar{c}}(0)| \leq |c(0) - \bar{c}(0)| \cdot \max \left\{ 1, 2 \frac{\beta + 1}{\epsilon} \right\} .$$

Proof: Abbreviate $h := c(0) - \bar{c}(0)$. $\Delta_h(t) := c(t) - \bar{c}(t)$,
 $\dot{\Delta}_h = \frac{d}{dt} \Delta_h$, $\ddot{\Delta}_h = \frac{d^2}{dt^2} \Delta_h$. From Taylor's theorem

$$|\Delta_h(\epsilon) - \Delta_h(0) - \epsilon \dot{\Delta}_h(0)| \leq \frac{\epsilon^2}{2} \max \{ |\ddot{\Delta}_h(t)| / 0 \leq t \leq \epsilon \} ,$$

whence

$$(4.26) \quad \epsilon |\dot{\Delta}_h(0)| \leq (1 + \beta) |\Delta_h(0)| + \frac{\epsilon^2}{2} \max \{ |\ddot{\Delta}_h(t)| / 0 \leq t \leq \epsilon \} .$$

Now the Lipschitz continuity of the geodesic differential equation of $\bar{g}(\cdot)$ with Lipschitz constant L gives with

$$\left\| \begin{pmatrix} \Delta_h(t) \\ \ddot{\Delta}_h(t) \end{pmatrix} \right\| := |\Delta_h(t)| + |\ddot{\Delta}_h(t)|$$

$$\begin{aligned}
 |\ddot{\Delta}_h(t)| &\leq \left| \begin{pmatrix} \dot{\Delta}_h(t) \\ \ddot{\Delta}_h(t) \end{pmatrix} \right| \leq \left| \bar{g} \begin{pmatrix} c(t) - \Delta_h(t) \\ \dot{c}(t) - \dot{\Delta}_h(t) \end{pmatrix} - \bar{g} \begin{pmatrix} c(t) \\ \dot{c}(t) \end{pmatrix} \right| \leq \\
 &\quad (4.26') \\
 &\leq L (|\Delta_h(t)| + |\dot{\Delta}_h(t)|) \leq L (|h| + |\dot{\Delta}_h(0)|) e^{L \cdot t}
 \end{aligned}$$

where the last inequality (4.26') is a well known estimation for the difference of solutions using the norm of the difference of their initial values.

Hence we have

$$(4.27) \quad |\ddot{\Delta}_h(t)| \leq L (|h| + |\dot{\Delta}_h(0)|) e^{L t}$$

We apply (4.27) to estimate $|\ddot{\Delta}_h(s)|$ in (4.26) and get

$$(4.28) \quad |\epsilon \dot{\Delta}_h(0)| \leq (\beta + 1) |h| + L (|h| + |\dot{\Delta}_h(0)|) e^{L \epsilon} \cdot \frac{1}{2} \epsilon^2$$

If $|\dot{\Delta}_h(0)| = 0$ then (4.25) holds trivially. If $|\dot{\Delta}_h(0)| \neq 0$ we get from (4.28) $|h| \neq 0$ because $L \cdot e^{L \epsilon} \cdot \frac{1}{2} \epsilon < 1$.

hence in this case we can write $|\dot{\Delta}_h(0)| = K|h|$. Inserting this in (4.28) yields

$$(4.29) \quad \epsilon K|h| \leq (\beta + 1) |h| + \frac{1}{2} \epsilon K|h| L (1 + 1) e^{L \epsilon} \cdot \epsilon \quad \text{if } K \geq 1,$$

in case $K \leq 1$ we are done. Now we have $L 2e^{L \epsilon} \cdot \epsilon \leq 1$.

Therefore (4.29) yields

$$\epsilon \cdot K|h| \leq (\tilde{\beta}(\epsilon) + 1) |h| + \frac{1}{2} \epsilon K|h| \quad \text{and thus } \frac{1}{2} \epsilon K \leq \beta + 1.$$

This gives $K \leq \frac{2(\beta + 1)}{\epsilon}$ and completes the proof of (4.25).

It is obvious that already the combination of Lemma 4.1 and Lemma 4.2 yields immediately the result "an extender is a Lipschitz point". Thus we have now one part of theorem 4.1 i.e. the subsequent.

Corollary 4.4: Let $q_0 \in M$. Let $r > 0$ be such that $B_r(q_0) \subset M \setminus \partial M$ and that $B_r(q_0)$ is contained in a domain of normal coordinates centered at q_0 . Then we have a positive number ρ and for every $\epsilon \in]0, \rho]$ there exists a number $Q(\epsilon) > 0$ such that the following holds:

If $A \subset M$ is closed and $q \in B_r(q_0)$ ¹⁾ is an ϵ -extender with respect to A and $d(q, A) \geq \frac{3}{2} \epsilon$, then

$$|\dot{c}_q - \dot{c}_{\bar{q}}| \leq Q(\epsilon) d(q, \bar{q})$$

for all $\bar{q} \in B_{\frac{\epsilon}{2}}(q)$.

Here $\dot{c}_q, \dot{c}_{\bar{q}}$ denote initial vectors of normalized minimal joins $c_q(t), c_{\bar{q}}(t)$ from q, \bar{q} to A respectively and $|\cdot|$ refers to the norm induced by Riemannian normal coordinates around q_0 .

1) Note $B_r(q_0) = \{q' \in M / d(q_0, q') \leq r\}$.

Proof of Corollary 4.4.:

Let $B_r(q_0)$ be contained in a domain $B_{R_1}(q_0)$ of Riemannian normal coordinates contained in $M \setminus \partial M$. Take $R \in]r, R_1[$. Apply Lemma 4.2 with $\bar{B} := B_R(q_0)$. If L is the Lipschitz constant with respect to the Riemannian normal coordinates on \bar{B} , choose $\varepsilon_1 \in]0, R - r[$ such that $2 \varepsilon_1 L e^{\varepsilon_1 L} \leq 1$. Then

For any $\beta > 0$, $\varepsilon \in]0, \varepsilon_1[$ and geodesics $c, \bar{c} :]0, \varepsilon[\rightarrow M$ with $c(0), \bar{c}(0) \in B_r(q_0)$ the inequality

$$(1) \quad |c(\varepsilon) - \bar{c}(\varepsilon)| \leq \beta |c(0) - \bar{c}(0)|$$

implies

$$|\dot{c}(0) - \dot{\bar{c}}(0)| \leq \max \left\{ 1, 2 \frac{1 + \beta}{\varepsilon} \right\} \cdot |c(0) - \bar{c}(0)|$$

Apply Proposition 4.1. There exist $f, F > 0$ such that

$$(2) \quad f d(q, \bar{q}) \leq |q - \bar{q}| \leq F d(q, \bar{q}) \quad \text{for all } q, \bar{q} \in B_R(q_0).$$

By Lemma 4.1 there exist $\varepsilon_0 > 0$ such that

For every $\varepsilon \in]0, \varepsilon_0[$ there exists a $\beta > 0$ such that:

$$(3) \quad \text{If } A \subset M \text{ is closed, } q \in B_r(q_0) \text{ is an } \varepsilon\text{-extender} \\ d(q, A) \geq \frac{3}{2} \varepsilon, \bar{q} \in B_{\frac{\varepsilon}{2}}(q), \text{ then} \\ d(c_q(\varepsilon), c_{\bar{q}}(\varepsilon)) \leq \frac{1}{\beta} d(q, \bar{q}).$$

Now put $\rho := \min\{\varepsilon_0, \varepsilon_1\}$. For $\varepsilon \in]0, \varepsilon_0[$ choose $\beta > 0$ as in (3), and put

$$Q(\varepsilon) := F \cdot \max \left\{ 1, 2 \frac{1 + \beta f}{\varepsilon \beta \bar{f}} \right\} .$$

If $A \subset M$ is closed, $q \in B_r(q_0)$ an ε -extender with respect to A , $\varepsilon \in]0, \rho[$, $d(q, A) \geq \frac{3}{2} \varepsilon$, $\bar{q} \in B_{\frac{\varepsilon}{2}}(q)$, then

$$\begin{aligned} |c_q(\varepsilon) - c_{\bar{q}}(\varepsilon)| &\stackrel{(3)}{<} \frac{1}{\beta} d(q, \bar{q}) \\ &\stackrel{(2)}{<} \frac{1}{\beta \bar{f}} |q - \bar{q}|, \end{aligned}$$

and hence by (1)

$$\begin{aligned} |\dot{c}_q - \dot{c}_{\bar{q}}| &\stackrel{(2)}{\leq} \max \left\{ 1, 2 \frac{1 + \frac{1}{\beta \bar{f}}}{\varepsilon} \right\} |q - \bar{q}| \\ &\stackrel{(2)}{<} \max \left\{ 1, 2 \frac{1 + \beta f}{\varepsilon \beta \bar{f}} \right\} F d(q, \bar{q}) \\ &= Q(\varepsilon) d(q, \bar{q}) . \end{aligned}$$

This proves Corollary 4.4.

Remark 4.3: In the above corollary $Q(\varepsilon)$ is a positive number depending on ε . It is not difficult to see that $Q(\varepsilon)$ can be chosen to be a continuous function of ε and such that $Q(\varepsilon)$ is monotone decreasing if ε is growing.

Remark 4.4: Note that we did neither claim nor prove in corollary 4.4 that an initial vector of some minimal join going from a point \bar{q} to A is uniquely determined by

that point $\bar{q} \neq q$, $d(q, \bar{q}) \leq \frac{\epsilon}{2}$. Indeed this need not be the case, see the examples described in figure 3.2 and figure 3.3.

Using corollary 4.4 we give in the following theorem a sharpened version of Lemma 4.1.

Theorem 4.5: Let D be a compact subset of (M, d) and assume there exists a number $\bar{s} > 0$ such that

$$D_{\bar{s}} := \{x \in M / d(x, D) \leq \bar{s}\} \subset M \setminus \partial M$$

Then there exists a number $\gamma > 0$ such that the subsequent statements hold:

a) We have a continuous function

$$]0, \infty[\times]0, \bar{s}[\ni (\epsilon, s) \mapsto \alpha(\epsilon, s) \in]0, \infty[$$

with the following property:

If $A \subset M$ is closed and $q \in D$ is an ϵ -extender relative to A then

$$\frac{d(q, \bar{q})}{d(c_q(s), c_{\bar{q}}(s))} \stackrel{1)}{\geq} \alpha(\epsilon, s)$$

for all \bar{q} with $d(q, \bar{q}) \leq \frac{1}{2} \min\{\gamma, \epsilon\}$ if

$$(B_{\frac{3}{2}\epsilon}(q) \cup c_q[0, s] \cup c_{\bar{q}}[0, s]) \cap A = \emptyset.$$

The function $\alpha(\epsilon, s)$ is monoton increasing in the variable ϵ and monoton decreasing if the variable s is growing.

1) $c_q(s)$, $c_{\bar{q}}(s)$ are normalized minimal joins from q, \bar{q} to A respectively.

b) Let $D_\gamma := \{x \in M / d(x, D) \leq \gamma\}$ and assume

$$D_\gamma \ni q \rightarrow u(q) \in [0, \bar{s}]$$

is a Lipschitz continuous function. Then we have a continuous function

$$]0, \infty[\times [0, \bar{s}] \ni (\varepsilon, u(q)) \rightarrow \Omega(\varepsilon, u(q)) \in]0, \infty[$$

with the following property:

If any point $q \in D$ is an ε -extender relative to A then

$$\frac{d(q, \bar{q})}{d(c_q(u(q)), c_{\bar{q}}(u(\bar{q})))} \geq \Omega(\varepsilon, u(q))$$

for all \bar{q} with $d(q, \bar{q}) \leq \frac{1}{2} \min\{\gamma, \varepsilon\}$ if

$$(B_{\frac{3}{2}\varepsilon}(q) \cup c_q[0, u(s)] \cup c_{\bar{q}}[0, u(\bar{q})]) \cap A = \emptyset.$$

The function $\Omega(\varepsilon, u(q))$ is monoton increasing in the variable ε and monoton decreasing if the variable $u(q)$ is growing

Remark 4.5: Theorem 4.5b) includes theorem 4.5a) as a special case namely take the function $u(\cdot)$ constant and equal to the number s . Nonetheless we think that theorem 4.5a) is of interest per se and we use the proof of theorem 4.5a) in the proof of theorem 4.5b).

Proof of Theorem 4.5a): Let

$$\exp_{q_i}^{-1} =: X_i : B_\delta(q_i) \rightarrow K_\delta(0) \subset T_{q_i} M,$$

be normal coordinates with center q_i and $B_\delta(q_i) \subset M \setminus \partial M$,

$K_\delta(0) := \{w \in T_{q_i} M / |w| \leq \delta\}$. We shall sometimes identify

$B_\delta(q_i)$ with $K_\delta(0)$. Let $\langle \cdot, \cdot \rangle$ be the Euclidean scalar

product in $T_{q_i} M$ and let e_1, \dots, e_n be an orthogonal basis in $T_{q_i} M$. We identify $T_{q_i} M$ with \mathbb{R}^n . Define

$$x_i^j(q) := \langle X_i(q), e_j \rangle$$

for all $q \in B_\delta(q_i)$ and every $j \in \{1, \dots, n\}$, $n := \dim M$.

The subsequent bijection gives obviously a trivialisation of $TB_\delta(q_i)$.

$$(K_\delta(0) \times \mathbb{R}^n) \ni (X_i(q), \sum_{j=1}^n v^j e_j) \xrightarrow{\sim} (q, \sum_{j=1}^n v^j \left(\frac{\partial}{\partial x_i^j}\right)_q) \in TB_\delta(q_i).$$

Thus we can define

$$\exp^i : (B_\delta(q_i) \times \mathbb{R}^n) \rightarrow \hat{M}$$

by

$$\exp^i(q, v) := \exp_q \left(\sum_{j=1}^n \langle v, e_j \rangle \left(\frac{\partial}{\partial x_i^j}\right)_q \right).$$

Clearly since $D_{\bar{s}} \subset M \setminus \partial M$ we have $d(D, \partial M) > \bar{s}$.

Let \bar{s}' be any number such that $d(D, \partial M) > \bar{s}' > \bar{s}$.

It is not difficult to see that for every $q_i \in D$ there exists a positive number δ_i such that

$$(4.31) \quad M \setminus \partial M \supset \exp^i(B_{\delta_i}(q_i) \times K_{\bar{s}}^-(0)) \stackrel{(***)}{\supset} \\ \{ \exp_q(w_q) \mid |w_q|_q \leq \bar{s}, q \in B_{\delta_i}(q_i) \} = \\ (*) \bigcup_{q \in B_{\delta_i}(q_i)} B_{\bar{s}}(q) \stackrel{(**)}{=} \{ \tilde{q} \in M \mid d(\tilde{q}, B_{\delta_i}(q_i)) \leq \bar{s} \}$$

Note (*) and (**) here are trivial equalities. Since

D is compact there exists a finite number of points

$$q_i, 1 \leq i \leq n \text{ such that } \bigcup_{1 \leq i \leq n} B_{\frac{1}{3}\delta_i}(q_i) \supset D.$$

We shall apply corollary 4.4. So as in corollary 4.4 there exists for every ball $B_{\delta_i}(q_i)$, $1 \leq i \leq n$ a number ρ_i . Define $\gamma := \min\{\delta_i, \rho_i / 1 \leq i \leq n\}$.

Let $q \in D$ be an ε -extender relative to A and assume $d(A, q) \geq \frac{3}{2}\varepsilon$. Clearly for some $i \in \{1, \dots, n\}$ the point $q \in B_{\frac{1}{3}\delta_i}(q_i)$. Define $\varepsilon' := \min\{\varepsilon, \gamma\}$. Then

by corollary 4.4

$$(4.32) \quad |\dot{c}_q - \dot{c}_{\bar{q}}|_i \leq Q_i(\varepsilon') d(q, \bar{q})$$

for all \bar{q} with $d(q, \bar{q}) \leq \frac{1}{2} \min\{\varepsilon, \gamma\}$, $|\cdot|_i$ the norm related to the normal coordinates with center q_i , and $\dot{c}_q, \dot{c}_{\bar{q}}$ initial vectors of minimal joins from q, \bar{q} to A respectively. Defining

$$Q(\varepsilon) := \max\{Q_i(\varepsilon') / 1 \leq i \leq n\}$$

it is easy to see that

$$]0, \infty[\ni \varepsilon \rightarrow Q(\varepsilon) \in]0, \infty[$$

is a continuous function and $Q(\varepsilon)$ is monoton decreasing if ε is growing because the function $Q_i(\varepsilon)$ have those properties, see remark 4.3. Therefore if q is an ε -extender relative to A then we have for some $i \in \{1, \dots, n\}$

$$(4.32') \quad |\dot{c}_q - \dot{c}_{\bar{q}}|_i \leq Q(\varepsilon) d(q, \bar{q})$$

for all \bar{q} with $d(q, \bar{q}) \leq \frac{1}{2} \min\{\gamma, \varepsilon\}$, if $d(q, A) \geq \frac{3}{2}\varepsilon$. By Lemma A.3 in the appendix there exists a number L_i such that

$$d(\exp^i(q, s\bar{v}), \exp^i(\bar{q}, \bar{v})) \leq L_i(d(q, \bar{q}) + |v - \bar{v}|_i)$$

holds for arbitrary $(q, v), (\bar{q}, \bar{v}) \in (B_{\delta_i}(q_i) \times K_{\mathbb{S}^1}(0))$.

Now if $v = \dot{c}_q^{-1}$ then

$$\exp^i(q, sv) = \exp_q(s \sum_{j=1}^n \langle v, e_j \rangle \left(\frac{\partial}{\partial x_j} \right)_q) = c_q(s).$$

Hence using (***) in (4.31)

$$d(c_q(s), c_{\bar{q}}(s)) \leq L_i (d(q, \bar{q}) + |\dot{c}_q - \dot{c}_{\bar{q}}|_i s)$$

if $0 \leq s \leq \bar{s}$. Therefore if $L := \max\{L_i / q \leq i \leq n\}$

then if $q \in D$ is an ϵ -extender (4.28) yields that

$$d(c_q(s), c_{\bar{q}}(s)) \leq L(1 + Q(\epsilon)s) d(q, \bar{q})$$

holds for all \bar{q} with $d(q, \bar{q}) \leq \frac{1}{2} \min\{\gamma, \epsilon\}$ and $0 \leq s \leq \bar{s}$.

This proves theorem 4.5a if we define

$$\alpha(\epsilon, s) := \frac{1}{L(1 + Q(\epsilon)s)}.$$

Proof of Theorem 4.5b: Using the notations from the above proof of theorem 4.5a) we get

$$\begin{aligned} \frac{d(q, \bar{q})}{d(c_q(u(q)), c_{\bar{q}}(u(\bar{q})))} &> \frac{d(q, \bar{q})}{d(c_q(u(q)), c_{\bar{q}}(u(q))) + d(c_{\bar{q}}(u(q)), c_{\bar{q}}(u(\bar{q})))} \\ &= \frac{d(q, \bar{q})}{d(c_q(u(q)), c_{\bar{q}}(u(q))) + |u(q) - u(\bar{q})|} \\ (4.33) \quad &> \frac{d(q, \bar{q})}{d(c_q(u(q)), c_{\bar{q}}(u(q))) + L_u d(q, \bar{q})} =: A \end{aligned}$$

with L_u being a Lipschitz constant of the function $q \mapsto u(q)$. Now theorem 4.5a yields

-
- 1) Recall as in corollary 4.4 and in (4.32) \dot{c}_q is identified with the representation of that vector relative to the normal coordinates with center q_i .

$$A \geq \begin{cases} \frac{1}{2} \alpha(\varepsilon, u(q)) & \text{if } L_u d(q, \bar{q}) \leq d(c_q(u(q)), c_{\bar{q}}(u(q))) \\ \frac{1}{2L} & \text{if } L_u d(q, \bar{q}) \leq d(c_q(u(q)), c_{\bar{q}}(u(q))) . \end{cases}$$

Therefore defining

$$\Omega(\varepsilon, u(q)) := \min \left\{ \frac{1}{2L}, \frac{1}{2} \alpha(\varepsilon, u(q)) \right\} ,$$

then (4.33) proves theorem 4.5b).

We have already proved one part of theorem 4.1 namely the result "an extender is a Lipschitz point", see corollary 4.4. Now we are going to treat the converse direction of this statement in theorem 4.1, i.e. we prove "a Lipschitz point is an extender". This result follows immediately from the subsequent

Lemma 4.3: We assume that the Riemannian metric on M has locally Lipschitz continuous derivatives. Let A be any closed subset of M . Let $q_0 \in M \setminus (\partial M \cup A)$. Then there exists $R > 0$ and for every $K > 0$ and $V \in]0, R]$ a positive number δ' such that the following holds:

- (i) There is a chart (X, U) , with $M \supset U \supset B_{3R}(q_0) := \{y \in M \mid d(y, q_0) \leq 3R\}$ and $X(U)$ is a convex and open set in \mathbb{R}^n .
- (ii) If $q \in B_R(q_0)$ and for all $\bar{q} \in B_V(q)$

$$(*) \quad |\dot{c}_q - \dot{c}_{\bar{q}}| \leq K d(q, \bar{q})$$

then q is a δ' -extender relative to A ; here $|\cdot|$ refers to the Euclidean norm related to the chart (X, U) and $\dot{c}_q, \dot{c}_{\bar{q}}$ are initial vectors of (non trivial) normalized minimal joins from q, \bar{q} to A .¹⁾

Proof of Lemma 4.3: The claim of (i) is trivial. By proposition 4.1 we have two numbers f, F such that

$$(4.34) \quad f \cdot d(q_1, q_2) \leq |q_1 - q_2| \leq F \cdot d(q_1, q_2)$$

holds for all $q_1, q_2 \in B_{3R}(q)$, $|\cdot|$ being the Euclidean norm related to the chart (X, U) . Further we know by proposition 4.1 that

1) We assume here that for all $\bar{q} \in B_V(q)$ there exist non trivial minimal joins from \bar{q} to A .

there exists a number L being a Lipschitz constant of the first order differential equation $\bar{g}(\cdot)$ for the geodesics in $B_{3R}(q)$. The differential equation $\bar{g}(\cdot)$ is a first order system which is given in local coordinates respective the chart (X,U) , see proposition 4.1. Now let q be any point in $B_R(q)$. We claim that the point q is a δ' -extender, with

$$(4.35) \quad \delta' := \left(\frac{1}{3} \min \{\bar{\delta}, V, R\}\right) > 0,$$

if we define $\bar{\delta} > 0$ by the equation

$$(4.36) \quad \left(1 + \frac{K}{\bar{\delta}}\right) e^{L \bar{\delta}} \cdot \bar{\delta} = 1.$$

Let $\delta \in]0, \delta']$ and denote the normalized geodesic backward extension of $c_q(t)$ by $\tilde{c}(s)$. This means $\tilde{c}(\delta) := q$ and $\tilde{c}(s + \delta) := c_q(s)$ for all $0 \leq s \leq d(A,q)$. We have to prove that $\tilde{c}(s)$ with $[0, d(A,q) + \delta] \ni s \rightarrow \tilde{c}(s) \in M$ is a minimal join from $\tilde{c}(0) =: x$ to the set A . For this let $c_x(s)$ be any normalised minimal join going from the point x to the set A . Because of technical convenience in our subsequent considerations we introduce the following mappings

$$\phi: [0, \delta] \ni s \rightarrow \phi(s) := \tilde{c}(\delta - s)$$

$$\psi: [0, \delta] \ni s \rightarrow \psi(s) := c_x(\delta - s),$$

$$\text{thus } \phi(0) = q, \phi(\delta) = \psi(\delta) = x.$$

If now $|\phi(0) - \psi(0)| = |\phi(0) - q| = 0$ we are done, since in this case the length of the minimal join c_x equals the length of the geodesic extension $\tilde{c}(\cdot)$, because here those subpaths of c_x, \tilde{c} which join q with A are both minimal joins, thus both have equal length $d(A,q)$. Therefore let us assume $|\phi(0) - \psi(0)| = |q - \psi(0)| \neq 0$. Further let $c_{\psi(0)}$ be that subpath of c_x such that $c_{\psi(0)}$ is a minimal join from $\psi(0)$ to A thus

$-\dot{\psi}(0) := -\frac{d}{ds}\psi(s)$ equals the tangent vector $\dot{c}_{\psi(0)}$ of $c_{\psi(0)}$ at

the point $\psi(0)$. We also denote $\dot{\phi}(s) := \frac{d}{ds}\phi(s)$ and have analogous

to the preceding consideration that $-\dot{\phi}(0) = \dot{c}_{\phi(0)} = \dot{c}_q$. Now since $c_{\psi(0)}$ is a minimal join from $\psi(0)$ to A with

$$(4.37) \quad d(q, \psi(0)) = d(\phi(0), \psi(0)) \leq 2\delta \leq \frac{2}{3} \min\{\bar{\delta}, V, R\}$$

(*) and (4.34) that

$$(4.38) \quad |\dot{c}_q - \dot{c}_{\psi(0)}| = |\dot{\phi}(0) - \dot{\psi}(0)| \stackrel{(+)}{\leq} K \cdot d(q, \psi(0)) \leq \frac{K}{f} |\phi(0) - \psi(0)|$$

where (+) holds because of the left hand side of (4.34). From (4.38) we get

$$(4.38') \quad |\dot{\phi}(0) - \dot{\psi}(0)| \leq \frac{K}{f} |\phi(0) - \psi(0)|$$

We use in the sequel estimates similar to those in the proof of lemma 4.2 (see also proposition 4.1, to clarify the background). First abbreviating $\frac{K}{f} =: \tilde{K}$ we get by (4.38') that

$$(4.39) \quad \left| \begin{pmatrix} \phi(0) \\ \dot{\phi}(0) \end{pmatrix} - \begin{pmatrix} \psi(0) \\ \dot{\psi}(0) \end{pmatrix} \right| := |\phi(0) - \psi(0)| + |\dot{\phi}(0) - \dot{\psi}(0)| \leq$$

$$\leq (1 + \tilde{K}) \cdot |\phi(0) - \psi(0)|$$

Now since the differential equation $g(\cdot)$ of the geodesics fulfills a Lipschitz condition with Lipschitz constant L we get using (4.39), (4.35), (4.37) that

$$|\phi(s) - \psi(s)| + |\dot{\phi}(s) - \dot{\psi}(s)| =: \left| \begin{pmatrix} \phi(s) \\ \dot{\phi}(s) \end{pmatrix} - \begin{pmatrix} \psi(s) \\ \dot{\psi}(s) \end{pmatrix} \right| \leq$$

$$\leq (1 + \tilde{K}) \cdot |\phi(0) - \psi(0)| e^{Ls} \quad \text{with say } 0 \leq s \leq \frac{R}{3}, \text{ see also}$$

inequality (4.26') in the proof of lemma 4.2. Thus

$$|\dot{\phi}(s) - \dot{\psi}(s)| \leq (1 + \tilde{K}) \cdot |\phi(0) - \psi(0)| e^{Ls} \quad \text{with } 0 \leq s \leq \delta. \text{ Hence}$$

$$|\phi(\delta) - \psi(\delta)| \geq |\phi(0) - \psi(0)| - \int_0^{\delta} |\dot{\phi}(t) - \dot{\psi}(t)| dt \geq$$

$$\geq |\phi(0) - \psi(0)| - (1 + \tilde{K}) \cdot |\phi(0) - \psi(0)| e^{L\delta} \cdot \delta$$

This gives

$$(4.40) \quad |\phi(\delta) - \psi(\delta)| \geq (|\phi(0) - \psi(0)| \frac{2}{3}) > 0 \quad \text{since}$$

$$|1 + \tilde{K}| e^{L\delta} \cdot \delta \leq \frac{1}{3} \quad \text{because of (4.36) using } 0 < \delta \leq \delta' \leq \frac{\bar{\delta}}{3}$$

and using the definition of K . However (4.40) yields a contradiction against $\phi(\delta) = \psi(\delta) = x$, thus $|\phi(0) - \psi(0)|$ must be zero, which proves lemma 4.3.

Remark 4.5: a) Note we did not require in Lemma 4.3 that the initial vectors of the normalized minimal joins going from q , \bar{q} to A are uniquely determined by the points $\bar{q} \in B_V(q)$. Although this uniqueness holds by (*) automatically at the point q , this uniqueness need not to hold at points $\bar{q} \in (B_V(q) \setminus \{q\})$ see e.g. figure 3.2 and figure 3.3.

b) The size of b) is of little significance for the (optimal) size of δ . If M is a flat two-dimensional cylinder, A a generator, then C_A is the "opposite" generator. Every minimal join from $q \in M \setminus (A \cup C_A)$ to A is a δ -extender, where $\delta = d(q, C_A)$ can be very small. On the other hand, when using isometric coordinates, $K = 0$.

Corollary 4.5: Let D be any compact set and assume $D \subset M \setminus \partial M$. Let A be any closed subset of M , then there exists a number $r > 0$ and for every $(K, V) \in]0, \infty[\times]0, r]$ a $\delta > 0$ such that for all $q_i \in D$ the following holds:

(a) $B_{3r}(q_i)$ is a domain of Riemannian normal coordinates around q_i and $B_{3r}(q_i) \subset M \setminus \partial M$.

(b) If $q \in B_r(q_i)$ and if for all $\bar{q} \in B_V(q)$

$$|\dot{c}_q - \dot{c}_{\bar{q}}|_i \leq K d(q, \bar{q})$$

then q is a δ -extender relative to A . Here $|\cdot|_i$ denotes the norm related to Riemannian normal coordinates with center q_i and $\dot{c}_q, \dot{c}_{\bar{q}}$ denote initial vectors of non trivial normalized minimal joins from q, \bar{q} to A respectively. 1)

Proof of corollary 4.5: For every point $q \in D$ there exists a number $r_q > 0$ such that $B_{3r_q}(q) \subset M \setminus \partial M$ and $B_{3r_q}(q)$ is contained in a domain of Riemannian normal coordinates with center q . Since D is compact there exists a set of say m points

$\{q_1, \dots, q_m\} =: F \subset D$ such that

$$D \subset \bigcup_{\ell=1}^m B_{\frac{1}{2}r_\ell}(q_\ell) \subset M \setminus \partial M,$$

with $r_\ell := r_{q_\ell}$. Using the continuity of the injectivity radius (see theorem 5.5 in § 5) it is obvious that there is a number

1) We assume here that for all $\bar{q} \in B_V(q)$ there exist non trivial minimal joins from \bar{q} to A .

$\tilde{r} > 0$ such that for all $q \in D$ $B_{3\tilde{r}}(q)$ is contained in a domain of normal coordinates with center q and $B_{3r}(q) \subset M \setminus \partial M$.

Now define

$$r := \min \left\{ \tilde{r}, \frac{1}{2} \min \{ r_1, \dots, r_m \} \right\}$$

then for all $q_i \in D$ $B_{3r}(q_i)$ is contained in a domain of normal coordinates around q_i and $B_{3r}(q_i) \subset M \setminus \partial M$. This proves (a).

We proceed now to prove (b). We shall apply Lemma 4.3. By Lemma 4.3 there exists for every point $q_\ell \in F$ a positive valued function $\delta'_\ell(\tilde{K}, V)$, $(\tilde{K}, V) \in]0, \infty[\times]0, r]$ with the following property:

(4.41) Every point $q \in B_r(q_\ell)$ is a $\delta'_\ell(\tilde{K}, V)$ -extender if

$$|\dot{c}_q - \dot{c}_{\bar{q}}|_\ell \leq \tilde{K} d(q, \bar{q})$$

holds for all $\bar{q} \in B_V(q)$; $|\cdot|_\ell$ denotes the norm related to normal coordinates with center q_ℓ .

Define

$$(4.41a) \quad \bar{\delta}(\tilde{K}, V) := \min \{ \delta'_\ell(\tilde{K}, V) \mid 1 \leq \ell \leq m \}.$$

We shall apply Lemma A.5 from the appendix, too. By Lemma A.5 there exist for every $\ell \in \{1, \dots, m\}$ two numbers L_ℓ, \bar{G}_ℓ such that for every point

$q_j \in B_{\frac{1}{2}r_\ell}(q_\ell)$ and for any two points $q, \bar{q} \in B_{2r}(q_j)$

$$(4.42) \quad |\dot{c}_q - \dot{c}_{\bar{q}}|_\ell \leq L_\ell |\dot{c}_q - \dot{c}_{q_j}| + \bar{G}_\ell d(q, \bar{q})$$

Here $\dot{c}_q, \dot{c}_{\bar{q}}$ are initial vectors of normalized paths

$c_q(t), c_{\bar{q}}(t), c_q(0) = q, c_{\bar{q}}(0) = \bar{q}$ and $|\cdot|_\ell, |\cdot|_j$ the norms related to Riemannian normal coordinates with center q_ℓ, q_j respectively. Define

$$(4.42a) \quad L := \min \{L_\ell \mid 1 \leq \ell \leq m\}$$

$$\bar{G} := \min \{G_\ell \mid 1 \leq \ell \leq m\}.$$

Now let q_i be any point in D and let q be a point in $B_r(q_i)$.

Assume that there exists $V \in]0, r]$ such that for all $\bar{q} \in B_V(q)$

$$|\dot{c}_q - \dot{c}_{\bar{q}}|_i \leq K d(q, \bar{q}),$$

$| \cdot |_i$ the norm related to normal coordinates with center q_i .

Clearly there exists a point $q_\ell \in F$ such that $q_i \in B_{\frac{1}{2}r_\ell}(q_\ell)$.

Hence $q \in B_{\frac{1}{2}r_\ell}(q_\ell)$ because $r \leq \frac{1}{2}r_\ell$. Therefore

using (4.42) and (4.42a) we have

$$|\dot{c}_q - \dot{c}_{\bar{q}}|_\ell \leq KL d(q, \bar{q}) + \bar{G} d(q, \bar{q})$$

for all $\bar{q} \in B_V(q)$. Thus

$$|\dot{c}_{\bar{q}} - \dot{c}_q|_\ell \leq (KL + \bar{G}) d(q, \bar{q})$$

for all $\bar{q} \in B_V(q)$. Therefore by (4.41) and (4.41a) there exists a positive number

$$\delta := \bar{\delta}((KL + \bar{G}), V)$$

such that q is a δ -extender. This proves (b) and completes the proof of corollary 4.5.

The following result illustrates theorem 4.5, it yields a criterion for a point to be a non-extender or equivalently a non-Lip-schitz point. Indeed using similar considerations like in the proof of theorem 4.5.a) it is now easy to prove.¹⁾

1) One has to apply arguments like in the proof of theorem 4.5.a) in a neighbourhood of the geodesic segment $c_q[0, s_0]$ below.

Theorem 4.6: Let A be any closed subset of (M, d) ,

let q be any point in $M \setminus (A \cup \partial M)$ and let q_n be a sequence of points with $\lim d(q_n, q) = 0$. If there exists some $s_0 > 0$ such that

$$(4.43) \quad c_q[0, s_0] \subset M \setminus (\partial M \cup A) \quad \text{and}$$

$$(4.47) \quad \lim_{n \rightarrow \infty} \frac{d(q_n, q)}{d(c_q(s_0), c_{q_n}(s_0))} = 0$$

$c_q(s)$, $c_{q_n}(s)$ normalized minimal joins from q , q_n to A , then q must be a non-extender relative to A . Thus q is a non-Lipschitz point by Lemma 4.3.

We discuss now the preceding result. Note in case we omit condition (4.43) in the assumption of theorem 4.6 then it is possible that q is an extender relative to A and (4.47) holds, too. In this case we have necessarily

$$(j) \quad c_q([0, s_0]) \cap (\partial M \cup A) \neq \emptyset.$$

Here now it may happen that $c_q([0, s_0]) \cap (\partial M \cup A) = A$. Namely take e.g. for M the Euclidean plane, let A contain only the origin 0 and let q_n be a sequence of points on the unit circle, q_n converging against a point q . Now if we choose $s_0 = 1$, we have the wanted situation with $c_q([0, 1]) \cap A = \{c_q(1)\} = \{0\} = A$.

If we assume $\partial M \cap c_q([0, s_0]) = \emptyset$ then (j) implies $c_q([0, s_0]) \cap A \neq \emptyset$.

In this case here we get $c_q([0, s_0]) \cap A = \{c_q(s_0)\}$ because $c_q(s)$ is a normalized minimal join from q to the set A .

One, short glance at figure 3.1 or figure 3.4 will convince the reader that even in case A is a point it is not hard to give examples for the following situation

(j') "Condition (4.44) holds with q being an extender while

$$c_q([0, s_0]) \cap A = \emptyset.$$

Clearly we have by theorem 4.6 that (j') implies $c_q([0, s_0]) \cap M \neq \emptyset$.

Here it is natural to ask whether (j') also implies that

$c_q([0, s_0]) \cap M$ contains a branching point relative to A , see definition 3.3.

We are inclined to think that the answer to this question is yes.

We started §4 with the reference to a classical result for unbordered manifolds. That result can in our terminology be also expressed in the following

Theorem 4.7': Let the point q be a non-extender relative to some point p . Then q must be a conjugate point relative to p if q is not a pica relative to p .

A proof of this result exploiting directly the geometric fact that q is a non-extender can e.g. be found in [45]p.97. We show here for an unbordered manifold the following

Theorem 4.7: Let q be a non-Lipschitz point relative to some point p . Then q must be a conjugate point relative to p if q is not a pica relative to p .

In our subsequent proof of theorem 4.7 we do not use the geometric non-extender property of the point q but use directly the non-Lipschitz point property of the point q and employ the contraposition of the mere analytic lemma 4.2. Of course theorem 4.7 implies within our setting the classical theorem 4.7', since we know that a non-extender is a non-Lipschitz point because we proved in lemma 4.3 the contraposition of this implication.

Proof: Since q is a non-Lipschitz point, we have a chart around q and a sequence q_n such that

$$(4.45) \quad |\dot{c}_q(0) - \dot{c}_{q_n}(0)| \geq n \cdot d(q, q_n), \quad n \in \mathbb{N}$$

with $\lim q_n = q$, $\dot{c}_{q_n}(0)$, $\dot{c}_q(0)$ initial vectors of minimal joins from q , q_n to p , $|\cdot|$ the norm related to the chart. Further since q is not a pica relative to p , we have $\lim \dot{c}_{q_n}(0) = \dot{c}_q(0)$. Therefore we get for the endvectors of the normalized minimal joins $c_{q_n}(t)$, $c_q(t)$ that

$$(4.46) \quad \lim \dot{c}_{q_n}(d(q_n, p)) = \dot{c}_q(d(q, p)) \in T_p M,$$

$$\dot{c}_{q_n}(t) = \frac{d}{dt} c_{q_n}(t)$$

Using Lemma 4.2 it is easily seen ¹⁾ that (4.45) yields the existence of a number R such that

$$(4.47) \quad |c_q(\epsilon) - c_{q_n}(\epsilon)| \geq R \cdot n \cdot |q - q_n|$$

holds for sufficiently small $\epsilon > 0$ and large enough natural numbers $n \in \mathbb{N}$. Now lift the geodesics $c_{q_n}(t)$ via \exp_p to the tangentspace $T_p M$.

Denote the lifted geodesics by $\tilde{c}_{q_n}(t)$; note we have here

$\exp_p(\tilde{c}_{q_n}(t)) = c_{q_n}(t)$. Now if $\tilde{q} = \tilde{c}_{q_n}(0)$ is not a critical point of the exponential map \exp_p , then \exp_p gives a bi-Lipschitz homeomorphism of a neighbourhood \tilde{U} of \tilde{q} onto a neighbourhood U of the point q . There-

fore (4.46) implies for sufficiently small $\epsilon > 0$ and large enough numbers $n \in \mathbb{N}$ that $\tilde{c}_{q_n}([0, \epsilon])$, $\tilde{c}_q([0, \epsilon]) \subset \tilde{U}$. By elementary plane geometry (say e.g. the Strahlensatz) we have in $T_p M \cap \tilde{U}$

$$(4.48) \quad |\tilde{c}_{q_n}(\epsilon) - \tilde{c}_q(\epsilon)|_p \leq |\tilde{q} - \tilde{q}_n|_p, \quad |\cdot|_p \text{ the norm in } T_p M.$$

Now (4.48) yields a contradiction against (4.47) because of the bi-Lipschitz homeomorphy of the map $\exp_p/\tilde{U}: (U, |\cdot|_p) \rightarrow (U, |\cdot|)$. Therefore \tilde{q} can not be a regular point of \exp_p . This proves theorem 4.7.

We proceed with a discussion on conjugate points in the cut locus.

Of course there may also exist picas in the cut locus C_A which are conjugate points, e.g. the north pole on the standard two sphere

if one defines A to be the one point set containing the south pole. -

Note we use the expression conjugate point also in relation to submanifolds or in relation to arbitrary closed sets. We always mean by a conjugate point relative to a set A either a singular value or say more general a substitute for a singular value of the exponential map relative to A , where this exponential map itself need not exist.¹⁾ So one might say more precisely singular point relative to A instead of conjugate point. However we will not be pedantic in this terminology question, since we believe that our intentions are always clear from the context. - The following definition seems to describe the desired intuitive content in order to characterize a conjugate point in the cut locus C_A in case A is a closed set and ∂M is not necessarily empty.

Definition 4.3: The point $q \in (M \setminus \partial M)$ is a conjugate point in the cut locus C_A , if there exists a sequence of points q_n , $\lim q_n = q$ and a positive number s_0 such that the following conditions hold: We

have

$$(4.49) \quad \lim \frac{d(q_n, q)}{|\dot{c}_q - \dot{c}_{q_n}|} = 0 \quad \text{and}$$

$$(4.50) \quad \text{for some } \delta \in]0, s_0[\text{ is } c_{q_i}([\delta, s_0]) \cap c_{q_j}([\delta, s_0]) = \emptyset$$

for all $i, j \in \mathbb{N}$ with $i \neq j$, $c_{q_n}(s)$, $c_q(s)$ normalized minimal joins from q , q_n to A and $c_q([0, s_0]) \subset M \setminus (\partial M \cup A)$, $|\cdot|$ the norm related to some appropriate chart.

1) See definition 4.3 below.

Remark 4.7: Condition (4.49) in the preceding definition can be replaced by the intrinsically described condition

$$\lim \frac{d(q, q_n)}{d(c_q(s_0), c_{q_n}(s_0))} = 0, \text{ because of theorem 4.6. - It is not}$$

difficult to see that definition 4.3 really describes a 'classical conjugate point' in the cut locus in the special case that A is a smooth submanifold of an unbordered complete Riemannian manifold.

We omit a proof of this statement since it is irrelevant to the sequel.

Suffice it to say that the proof being technically similar to the proof of theorem 4.7 uses the fact that there exists a subsequence of the segments $(c_{q_n}[\delta, s_0])_n$ which converges against the segment

$c_q([\delta, s_0])$. - Note in case $\partial M \neq \emptyset$, exploiting the branching behaviour of minimal joins, one can easily give examples where for some $s > 0$

the sequence $(c_{q_j}([0, s]))_j$ does not contain any subsequence converging against the segment $c_q([0, s])$, while $\lim q_n = q$ and $\lim \dot{c}_{q_n} = \dot{c}_q$.

This situation is obviously impossible in case $\partial M = \emptyset$. Of course in this context it is natural to ask how to define an arbitrary conjugate point relative to any closed set A in a bordered manifold.

One has to modify definition 4.3 for this purpose. In this case the paths $c_{q_n}(t)$, $c_q(t)$ in definition 4.3 are not any longer necessarily minimal joins but geodesics from q , q_n to the set A. Clearly also in this situation condition (4.49) and (4.51) should hold. However, one also has to impose a condition on the lengths of the geodesics

$c_{q_n}(t), c_q(t)$ say at least that the sequence

$t_{q_n} := \min\{s \mid \text{geodesic } c_{q_n}([0,s]) \cap A \neq \emptyset\}$ converges against t_q .

Otherwise geodesic spirals which are dense on the flat torus give material to construct examples where the description by (4.49) and (4.50) alone does not agree with the classical definitions in case $\partial M = \emptyset$ and A a submanifold or even a one point set in M .

§5 APPLICATIONS

In this paragraph I want to apply results of §3 and §4 in order to derive several theorems. Some of these theorems are well-known. However giving new proofs for these theorems I want to show that the technics used in §3 and §4 which were originally developed to investigate cut loci in bordered manifolds give also a common frame for results of Jacobi, Bangert, Federer, Kleinjohann and R. Walter which belong to apparently different topics. During the whole paragraph unless we say anything else let (M,d) be a space as assumed in corollary 4.4. Let A be any closed set in M . The combination of theorem 3.1 and corollary 4.4 yields immediately.

Theorem 5.1: The gradient of the distance function $d(A, \cdot)$ is locally Lipschitz continuous on $M \setminus (C_A \cup \partial M \cup A)$.

Remark 5.1: This theorem makes it possible to apply the Gauss-Bonnet theorem to pieces of distance hypersurfaces relative to A which do not meet $C_A \cup \partial M \cup A$, see also [38] page 343 and [72] p.20-22¹⁾.

Using theorem 4.5a we give next a new proof of a well known theorem i.e. the subsequent theorem 5.2 a special case of which is a famous result of Jacobi, see also [16] page 231. All proofs known to us for theorem 5.2 as well as for Jacobis theorem use second variation and index form technics, while we do not use these methods here.

1) These authors prove and use the Gauss-Bonnet theorem under weak regularity assumptions. See in particular [38](3.7) Satz and cf. theorem 5.7 in this paragraph.

Theorem 5.2: Let S be a C^2 -smooth k -dimensional, closed submanifold of a complete unbordered n -dimensional C^∞ -smooth Riemannian manifold M , $k \in \{1, \dots, (n-1)\}$. Let $\perp S$ be the normal bundle of S in M . Let for some $s_0 > 0$ $\exp_S(s_0 N_p) = q$ be a singular value of the exponential map \exp_S defined on the normal bundle i.e. $\exp_S: \perp S \rightarrow M$, $\exp_S(0 N_p) = p$ being the footpoint of the unit normal vector N_p . Then the geodesic $g(s) := \exp_S(s N_p)$ is not any longer a locally¹⁾ minimal join to S for any number $s > s_0$.

Remark: Jacobis theorem differs from theorem 5.2 only in so far that the submanifold S in theorem 5.2 is replaced by a single point p . Thus we have here $\dim S = \dim \{p\} = 0$ and this violates our assumption $1 \leq \dim S \leq (n-1)$ in theorem 5.2. However theorem 5.2 yields easily Jacobis theorem if we take in this case as submanifold S instead of the point p a small distance sphere

$$S_r(p) := \{x \mid x \in M, d(p, x) = r\} .$$

Proof of theorem 5.2: The proof is performed in several steps. We start with a description of the geometric situation used in the proof and we introduce notations. Let us take a chart for a neighbourhood U of the ray $\{s N_p \mid s \geq 0\}$ in the normal bundle $\perp S$, with $\|\cdot\|_1$ being the norm related to this chart. We can assume that in this chart $U \cap S$ is an open subset of a k -dimensional plane H in the chart space and that all fibers of $\perp S$ with footpoints in $U \cap S$ are orthogonal to H in the chart space. Identifying $(U \cap \perp S)$ with the related subset in the chart space we can assume that the exponential map of the normal bundle i.e. $\exp_S: U \cap \perp S \rightarrow M$ maps segments orthogonal to H on geodesics of equal length.

1) This means every neighbourhood of the geodesic piece $g[0, s]$ contains paths joining S with $g(s)$ and those paths have a length shorter than s . See the remark after the end of the proof on page 122.

The point p corresponds to the zero vector in the chart space. Since S is C^2 -smooth we have that \exp_S is C^1 -smooth and this is sufficient for our subsequent considerations. According to the assumption of theorem 5.2 $\exp_S(s_0 N_p) = q$ is a singular value of the exponential map \exp_S . Clearly it is no restriction for theorem 5.2 if we assume that $\tilde{q} := s_0 N_p$ is the first critical point of \exp_S along the ray $\{s N_p | s \geq 0\}$. Now let v be a vector in the kernel of the differential $D\exp_S(\tilde{q})$ thus $(D\exp_S(\tilde{q}))(v) = 0$. We identify v with a vector in the chart space of $UN\mathcal{L}S$. Let $\tilde{c}_{q_v}(s)$ be the normalized Euclidean segment in $UN\mathcal{L}S$ which is contained in that ray of $\mathcal{L}S$ which joins $\tilde{q} + v$ with S , thus $\tilde{c}_{q_v}(0) = \tilde{q} + v$. We define $\tilde{c}_q(s) := \tilde{c}_{q_v}(s)$, thus $\tilde{c}_q(0) = \tilde{q}$. We introduce the following notations for the paths $\exp_S(\tilde{c}_{q_v}(s)) =: c_{q_v}(s)$, joining in $\exp_S(UN\mathcal{L}S)$ the points $q_v := \exp_S(\tilde{q} + v)$, $q := \exp_S(\tilde{q})$ with S by geodesics corresponding to rays in the normal bundle $UN\mathcal{L}S$. By proposition 4.1 we can choose a chart $(K(q), \cdot | \cdot)$ for a neighbourhood of q such that there exist positive numbers a, b with

$$(5.1) \quad a d(q_1, q_2) \leq |q_1 - q_2| \leq b d(q_1, q_2) \quad \text{for all } q_1, q_2 \in K(q),$$

$|\cdot|$ the norm related to the chart. We will use below for the geodesics in $K(q)$ the first order differential equation described in proposition 4.1, $L > 0$ being the related Lipschitz constant.

Now for a sufficiently small positive number t_1 we get

$$c_{q_v}(t) \in K(q) \quad \text{for all } t \in [0, t_1] \text{ and } v \text{ with } |v|_1 \leq t,$$

We use notations like in lemma 4.2 namely $q - q_v =: h_v$, $c_q(t) - c_{q_v}(t) =: \Delta_{h_v}(t)$, analogous $\frac{d}{dt} \Delta_{h_v}(t) =: \dot{\Delta}_{h_v}(t)$, $\frac{d^2}{dt^2} \Delta_{h_v}(t) =: \ddot{\Delta}_{h_v}(t)$.

For the proof of theorem 5.2 we proceed now in four steps which we show separately at the end of the proof.

Step 1: If we choose a positive number $t_2 \leq \min \{1, t_1\}$ such that $(\frac{1}{2} t_2 L e^{Lt_2}) < \frac{1}{2}$, then we have

$$(5.2) \quad 2|h_v| + |\Delta_{h_v}(f)| \geq \frac{1}{2} t |\dot{\Delta}_{h_v}(0)|$$

for all positive numbers $t \leq t_2$ and $|v|_1 \leq t_2$.

Step 2: There exist three positive numbers $R, \bar{r}, t_3; \bar{t} \leq t_2; t_3 \leq t_2$ such that

$$(5.3) \quad d(c_q(\bar{f}), c_{q_v}(\bar{f})) \geq R|v|_1$$

holds for all v with $|v|_1 \leq t_3$.

Step 3: Using the results proved in step 1 and step 2 we show there exist two positive valued functions

[kernel: $D \exp_S(q)(\cdot) \ni v \mapsto \delta(v), (0, \bar{f}) \ni \bar{f} \mapsto f(\bar{f})$]

with $\lim_{v \rightarrow 0} \delta(v) = 0, \lim_{\bar{f} \rightarrow 0} f(\bar{f}) = 0$ such that

$$\frac{d(c_q(\bar{f}), c_{q_v}(\bar{f}))}{d(c_q(\bar{f}), c_{q_v}(\bar{f}))} \leq \delta(v) + f(\bar{f})^1, \text{ for all } v \text{ with } |v|_1 < t_3.$$

Step 4: Let $\bar{f} \in (0, \bar{f})$. Assuming that $c_q(\bar{f})$ is an extender relative to S we find a positive number $\eta(\bar{f})$ such that every geodesic piece $c_{q_v}[\bar{f}, \bar{f} + d(S, c_{q_v}(\bar{f}))]$ is a minimal join from $c_{q_v}(\bar{f})$ to S , for any v with $|v|_1 \leq \eta(\bar{f})$.

Using the results got in step 3 and step 4 together with theorem 4.5a we prove now the claim of theorem 5.2. We argue by contradiction. For this let ϵ be any positive number and assume the point q is an ϵ -extender relative to S . We want

1) Those functions $\delta(v), f(\bar{f})$ will be technically useful in our proof.

to derive a contradiction. To this end choose $\tilde{\epsilon} \in (0, \bar{\epsilon})$ so small that $f(\tilde{\epsilon}) \leq \frac{1}{2} \alpha(\epsilon, \bar{\epsilon})$. This function $\alpha(\cdot)$ was explained in theorem 4.5a. Further choose $v \neq 0$ so small that both conditions $|v|_1 \leq \eta(\tilde{\epsilon})$ and $\delta(v) \leq \frac{1}{2} \alpha(\epsilon, \bar{\epsilon})$ hold. Now since q is an ϵ -extender relative to S the point $c_q(\tilde{\epsilon})$ is an ϵ -extender relative to S too. Using the result in step 4 together with the condition $|v|_1 \leq \eta(\tilde{\epsilon})$ we get that the geodesic pieces $c_{q_v}[\tilde{\epsilon}, \tilde{\epsilon} + d(S, c_{q_v}(\tilde{\epsilon}))]$ are minimal joins from $c_{q_v}(\tilde{\epsilon})$ to S . Therefore exploiting theorem 4.5a we get

$$\frac{d(c_q(\tilde{\epsilon}), c_{q_v}(\tilde{\epsilon}))}{d(c_q(\tilde{\epsilon} + (\bar{\epsilon} - \tilde{\epsilon})), c_{q_v}(\tilde{\epsilon} + (\bar{\epsilon} - \tilde{\epsilon})))} \geq \alpha(\epsilon, (\bar{\epsilon} - \tilde{\epsilon})) \geq \alpha(\epsilon, \bar{\epsilon}) > 0$$

However, according to our conditions for $\tilde{\epsilon}$ and $|v|_1$, we get from the result in step 3 that

$$\frac{d(c_q(\tilde{\epsilon}), c_{q_v}(\tilde{\epsilon}))}{d(c_q(\bar{\epsilon}), c_{q_v}(\bar{\epsilon}))} \leq \frac{1}{2} \alpha(\epsilon, \bar{\epsilon})$$

This is a contradiction.

Proof of the claim in step 1: Put $\Delta(s) := \Delta_{h,v}(s)$. Recall from (4.27) $|\ddot{\Delta}(s)| \leq L (|\Delta(0)| + |\dot{\Delta}(0)|) e^{Lt}$ for all $0 < s < t$.

Therefore, by Taylor's formula

$$\begin{aligned} |\Delta(t)| &\geq t |\dot{\Delta}(0)| - |\Delta(0)| - \frac{1}{2} t^2 L (|\Delta(0)| + |\dot{\Delta}(0)|) e^{Lt} \\ &= t |\dot{\Delta}(0)| \left(1 - \frac{1}{2} t L e^{Lt}\right) - |\Delta(0)| \left(1 + \frac{1}{2} t^2 e^{Lt}\right) \\ &\geq \frac{1}{2} t |\dot{\Delta}(0)| - 2 |\Delta(0)| \end{aligned}$$

for $0 < t < t_2$. This proves the claim in step 1.

Proof of the claim in step 2: In our chart $U \cap \mathbb{I}S$ the paths $\tilde{c}_{q_v}(t)$ and $\tilde{c}_q(t)$ yield Euclidean segments. Therefore the tangent-vector $\frac{d}{dt} \tilde{c}_{q_v}(t) = \dot{\tilde{c}}_{q_v}(t) = \dot{\tilde{c}}_{q_v}(0)$ is constant. This gives

$$\begin{aligned} |\tilde{c}_q(\bar{t}) - \tilde{c}_{q_v}(\bar{t})|_1 &= |\tilde{c}_q(0) + \int_0^{\bar{t}} \dot{\tilde{c}}_q(0) dt - (\tilde{c}_{q_v}(0) + \int_0^{\bar{t}} \dot{\tilde{c}}_{q_v}(0) dt)|_1 \\ &\geq |v|_1 - \bar{t} |\dot{\tilde{c}}_q(0) - \dot{\tilde{c}}_{q_v}(0)|_1 . \end{aligned}$$

In the Euclidean chart space the point \tilde{q} (being an extender relative to the plane H) is a Lipschitz point relative to H . Therefore we have a positive number β such that $|\dot{\tilde{c}}_q(0) - \dot{\tilde{c}}_{q_v}(0)|_1 \leq \beta |v|_1$. Therefore if we choose $\bar{t} \in (0, \frac{1}{2\beta}]$ we have

$$|\tilde{c}_q(\bar{t}) - \tilde{c}_{q_v}(\bar{t})| \geq |v|_1 - \bar{t} |\dot{\tilde{c}}_q(0) - \dot{\tilde{c}}_{q_v}(0)|_1 \geq |v|_1 - \bar{t} \beta |v|_1 \geq \frac{1}{2} |v|_1$$

for all v with $|v|_1 \leq t_2$. Now since $\tilde{c}_q(\bar{t})$ is no critical point of the exponential map $\exp_S: \mathbb{I}S \rightarrow M$, the map $\exp_S: \tilde{V} \rightarrow (\exp_S(\tilde{V})) := V$ is a bilipschitz homeomorphism on a neighbourhood \tilde{V} of $\tilde{c}_q(\bar{t})$. Clearly, because $\lim_{v \rightarrow 0} \tilde{c}_q(\bar{t}) = \tilde{c}_q(\bar{t})$, there exists a positive number $t_3 \leq t_2$ such that we have $\tilde{c}_{q_v}(\bar{t}) \in \tilde{V}$ for all v with $|v|_1 \leq t_3$. Using the bilipschitz homeomorphy of the map $\exp_S: \tilde{V} \rightarrow V$ we know there is a number R such that combining the facts above leads to $\frac{1}{2R} d(c_q(\bar{t}), c_{q_v}(\bar{t})) = \frac{1}{2R} d(\exp_S(\tilde{c}_q(\bar{t})), \exp_S(\tilde{c}_{q_v}(\bar{t}))) \geq |\tilde{c}_q(\bar{t}) - \tilde{c}_{q_v}(\bar{t})| > \frac{1}{2} |v|_1$. Thus $d(c_q(\bar{t}), c_{q_v}(\bar{t})) \geq R |v|_1$ for all v with $|v|_1 \leq t_3$. This proves our claim in step 2.

Proof of the claim in step 3:

Let $\tilde{t} \in (0, \bar{t})$, and let $|v|_1 \leq t_3$ then we get

$$\begin{aligned} |c_q(\tilde{t}) - c_{q_v}(\tilde{t})| &\leq |h_v| + \int_0^{\tilde{t}} |\dot{\Delta}_{h_v}(t)| dt \\ (5.4) \quad &\leq |h_v| + \tilde{t}(|h_v| + |\dot{\Delta}_{h_v}(0)|)e^{L\tilde{t}} \end{aligned}$$

$$(5.5) \quad \leq |h_v| + \tilde{t}(|h_v| + (2|h_v| + |\Delta_{h_v}(\bar{t})|) \frac{2}{\tilde{t}}) e^{L\tilde{t}}$$

$$(5.6) \quad \leq |h_v| (1 + (\frac{1}{2}\tilde{t} + 2)(\tilde{t} e^{L\tilde{t}} \frac{2}{\tilde{t}})) + (\tilde{t} e^{L\tilde{t}} \frac{2}{\tilde{t}}) |\Delta_{h_v}(\bar{t})|$$

where (5.4) holds because of (4.26') and (5.5) is got using (5.2).

We denote the right hand side of inequality (5.6) by (A). Now

defining a function $(\tilde{t}, \tilde{t}) \longmapsto \tilde{f}(\tilde{t}, \tilde{t}) := \tilde{t} e^{L\tilde{t}} \frac{2}{\tilde{t}}$

and using that $\tilde{t} \leq 1$ we get

$$(A) \leq |h_v| (1 + 3 \tilde{f}(\tilde{t}, \tilde{t})) + \tilde{f}(\tilde{t}, \tilde{t}) |\Delta_{h_v}(\bar{t})|.$$

Hence

$$(5.7) \quad \frac{|c_q(\tilde{t}) - c_{q_v}(\tilde{t})|}{|c_q(\tilde{t}) - c_{q_v}(\tilde{t})|} \leq \frac{|q - q_v|}{|c_q(\tilde{t}) - c_{q_v}(\tilde{t})|} (1 + 3 \tilde{f}(\tilde{t}, \tilde{t})) + \tilde{f}(\tilde{t}, \tilde{t})$$

Using (5.3) and (5.1) we get

$$(5.8) \quad |c_q(\tilde{t}) - c_{q_v}(\tilde{t})| \geq R \alpha |v|_1 \quad \text{with } |v|_1 \leq t_3$$

Exploiting (5.8) and (5.1) we get from (5.7)

$$(5.9) \quad \frac{d(c_q(\tilde{t}), c_{q_v}(\tilde{t}))}{d(c_q(\tilde{t}), c_{q_v}(\tilde{t}))} \leq \frac{|q - q_v|}{|v|_1} \frac{b}{R\alpha^2} (1 + 3 \tilde{f}(\tilde{t}, \tilde{t})) + \frac{b}{\alpha} \tilde{f}(\tilde{t}, \tilde{t})$$

Now we have $|q - q_v| = |\exp_s(\tilde{q}) - \exp_s(\tilde{q} + v)| = o(|v|_1)$ because

$D\exp_s(\tilde{q})(v) = 0$, $o(\cdot)$ the Landau 0-symbol. Therefore exists a

function $\delta(v)$, with $\lim_{v \rightarrow 0} \delta(v) = 0$ and

$$\frac{|q - q_v|}{|v|_1} \frac{b}{R\alpha^2} (1 + 3\bar{f}(\bar{f}, \bar{f})) \leq \delta(v) \quad \text{for all } \bar{f} \in (0, \bar{f}).$$

Using this and defining $f(\tilde{t}) := \frac{b}{\alpha} \bar{f}(\bar{t}, \tilde{t})$

inequality (5.9) yields
$$\frac{d(c_q(\tilde{f}), c_q(\bar{f}))}{d(c_q(\bar{f}), c_q(\bar{f}))} \leq \delta(v) + f(\bar{f}).$$

This proves the claim in step 3.

Proof of the claim in step 4:

By our assumptions the exponential map \exp_S has no singularity in an open neighbourhood \tilde{O} of $\tilde{c}_q[\tilde{t}, s_0]$. Therefore $\exp_S: \tilde{O} \rightarrow \exp_S(\tilde{O})=0$ is a local diffeomorphism on \tilde{O} .

Because of that we have a number $\gamma > 0$, such that for all $t^* \in [\tilde{t}, s_0]$ $\exp_S: \tilde{B}_\gamma(t^*) \rightarrow \exp_S(\tilde{B}_\gamma(t^*)) =: B_\gamma(t^*)$ is a diffeomorphism, $\tilde{B}_\gamma(t^*) := \{x \in U \cap \perp S \mid |x - \tilde{c}_q(t^*)|_1 \leq \gamma\} \subset \tilde{O}$.

Denote for all v with $|v|_1 \leq \frac{1}{3}\gamma$ minimal joins from $c_{q_v}(\tilde{t})$ to S by $t \mapsto g_v(t)$, $g_v(0) := c_{q_v}(\tilde{t})$, $g(t) := c_q(\tilde{t} + t)$. In general

$g_v[0, d(S, g_v(0))]$ need not to be contained in O . However $c_q(\tilde{t})$ being an extender is not a picard relative to S . Because of that we

have $\lim_{v \rightarrow 0} (g_v[0, d(g_v(0), S)]) = c_q[\tilde{t}, s_0]$. Therefore there exists a number $\eta(\tilde{t})$ such that $g_v[0, d(g_v(0), S)] \subset \cup_{t^* \in [\tilde{t}, s_0]} B_\gamma(t^*)$ for all v with $|v|_1 \leq \eta(\tilde{t})$

Using this we lift the geodesic piece $g_v[0, d(g_v(0), S)]$ via the map \exp_S to the normal bundle. We start to lift at the point $g_v(d(g_v(0), S))$ and find that the lifted geodesic piece called $\tilde{g}_v[0, d(g_v(0), S)]$ ends up in $\tilde{c}_{q_v}(\tilde{t})$ and for that $\tilde{g}_v[0, d(g_v(0), S)] = \tilde{c}_{q_v}[\tilde{f}, d(g_v(0), S) + \tilde{f}]$.

The last equality holds because there is only one segment in

$\tilde{O} \in U \cap \perp S$ having the property to start in $\tilde{c}_{q_v}(\tilde{t})$ and being normal on $H \cap \perp S \cap \tilde{O}$. Using $\tilde{c}_{q_v}[\tilde{t}, d(g_v(0), S) + \tilde{t}] = \tilde{g}_v[0, d(c_{q_v}(\tilde{t}), S)]$ we get $c_{q_v}[\tilde{t}, d(g_v(0), S) + \tilde{t}] = g_v[0, d(c_{q_v}(\tilde{t}), S)]$. Therefore $c_{q_v}[\tilde{t}, d(c_{q_v}(\tilde{t}), S) + \tilde{t}]$ is a minimal join from $c_{q_v}(\tilde{t})$ to S . Moreover this minimal join is uniquely determined by its initial point $g_v(0) = c_{q_v}(\tilde{t})$ if $|v|_1 \leq \eta(\tilde{t})$. This proves the claim in step 4.

Remark: All considerations used in the preceding proof and all arguments they are based on (i.e. crucially lemma 4.1) are essentially local ones. In particular we do not make any assumption for S outside of a neighbourhood of the point p . Therefore it is not difficult to see that $g[0, s]$ fails to be a locally minimal join to S for any number $s > s_0$. Namely given the conditions of theorem 5.2 and assuming that for some $\bar{\epsilon} > 0$ the path $g[0, s_0 + \bar{\epsilon}]$ is a locally minimal join from S to $g(s_0 + \bar{\epsilon})$ we can reformulate the result of step 4 replacing minimal by locally minimal and extender by local extender. Using this reformulated result and the result of step 3 with properly chosen $\tilde{t}, \bar{t} \in (0, \bar{\epsilon})$, $\tilde{t} < \bar{t}$, then for small enough $\delta(v) + f(\tilde{t})$ the construction in lemma 4.1, applied with central point $c_q(\tilde{t})$ and radius $\epsilon := \bar{t} - \tilde{t}$ yields a path b with length smaller than $s_0 - 2\tilde{t} + \bar{t}$. This path b joining S with the point $g(s_0 - 2\tilde{t} + \bar{t})$ is contained in an arbitrary small neighbourhood of $g[0, s_0 + \bar{\epsilon}]$.

Theorem 5.3: Let S be a C^2 -smooth, k -dimensional, closed submanifold of a complete, unbordered n -dimensional C^∞ -smooth Riemannian manifold M , $k \in \{0, \dots, (n-1)\}$. Let $(\perp S, \pi, S)$ be the normal bundle of S in M , $\pi: \perp S \rightarrow S$ being the projection of any vector $x \in \perp S$ on the footpoint $\pi(x) \in S$. Denote by $\perp S_1$

the subbundle of TS consisting of all vectors in TS which have length one i.e. $TS_1 = \{x \in TS \mid |x|_{\pi(x)} = 1\}$, $| \cdot |_{\pi(x)}$ the norm related to the metric in the tangent space $T_{\pi(x)}M$ of M at the point $\pi(x)$.

a) Then the map $s: TS_1 \rightarrow (0, \infty]$, $s(x) := \sup\{t/d(S, \exp_S(tx)) = t\}$ is continuous.

b) Let $\bar{s}: TS_1 \rightarrow \{x/s(x) < \infty\} \rightarrow M$, $x \mapsto \exp_S(s(x)x)$. Then the set $\bar{s}(TS_1)$ consisting of all non-extenders relative to S is closed. Therefore $\bar{s}(TS_1)$ is the cut locus C_S of S in M .

Proof of theorem 5.3:¹⁾ The statement $s(TS_1) \subset (0, \infty]$ holds mainly due to two facts. First for any zero vector $0_p \in TS$ exists a neighbourhood \tilde{U} in TS such that $\exp_{S, \tilde{U}}: \tilde{U} \rightarrow \exp_S(\tilde{U})$ is a diffeomorphism. Second:*) "Minimal joins from any point $p \in M$ to S are normal on S ". A proof for the second statement makes use of the following fact. Namely for any $p \in M$ there exists a positive number r such that the function $d(p, \cdot)$ is C^1 -smooth on $\{x \in M \mid 0 < d(x, p) < r\}$. Further using *) and the definition of the map s it is easy to see that $\bar{s}(TS_1)$ is the set of all non-extenders relative to S .

We now prove theorem 5.3b We show: **) "The set of all non-extenders relative to S is closed."

For this, let $(q_n) \in M$ be a sequence of non-extenders relative to S and assume (q_n) converges against a point $q_0 \in M$. We must prove that q_0 is a non-extender relative to S . Now by theorem 3.1 the picas are dense in the set of all non-extenders and we may therefore assume that the q_n are picas. Using this we prove now that q_0 is a non-extender. If q_0 is a

1) The ideas in the proof of theorem 5.3 are essentially classical ones. However, we give the proof here for the sake of completeness.

pica we are done. So we assume that q_0 is not a pica. Consequently let $c_{q_0} [0, d(q_0, S)]$ be the unique minimal join from q_0 to S , $q_0 = c_{q_0}(0)$, $\frac{d}{dt} c_{q_0}(t) = \dot{c}_{q_0}(t)$. Now, if q_0 is an extender then by theorem 5.2 the map \exp_S must be a diffeomorphism on an open neighbourhood \tilde{U} of the point $(-d(S, q_0), \dot{c}_{q_0}(d(S, q_0))) \in \mathbb{I}S$.

Let $(c_n), c(\tilde{c}_n)$ be sequences of minimal joins from q_n to S , $c_n \neq \tilde{c}_n$. Then $\lim \dot{c}_n(0) = \dot{c}_{q_0} = \lim \dot{\tilde{c}}_n(0)$, and therefore

$$\{-d(S, q_n), \dot{c}_n(d(S, q_n)), d(S, q_n), \dot{\tilde{c}}_n(d(S, q_n))\} \subset \tilde{U}$$

for sufficiently large n . This contradicts

$$\exp_S(-d(S, q_n), \dot{c}_n(d(S, q_n))) = q_n = \exp_S(-d(S, q_n), \dot{\tilde{c}}_n(d(S, q_n))) .$$

This proves theorem 5.3b.

We now prove theorem 5.3a. For this we shall prove the following implication: "If a sequence $(x_n) \in \mathbb{I}S$ converges against $x_0 \in \mathbb{I}S_1$ then $s(x_n)$ converges against $s(x_0) \in [0, \infty]$."

First we treat the case $s(x_0) < \infty$. We argue by contradiction. Namely let ϵ be any positive number and assume there exists a (here equally denoted) subsequence of $(s(x_n))$ with $s(x_n) \notin (s(x_0) - \epsilon, s(x_0) + \epsilon)$, for all $n \in \mathbb{N}$. Then at least one of the following two statements must hold.

- I) There exists a (here equally denoted) subsequence of $(s(x_n))$ with $0 < s(x_n) < s(x_0) - \epsilon$ for all $n \in \mathbb{N}$.
- II) There exists a (here equally denoted) subsequence of $(s(x_n))$ with $s(x_n) > s(x_0) + \epsilon$ for all $n \in \mathbb{N}$.

Let us consider first case I). Here $s(x_n)$ must have a cluster point

$r_0 \in [0, s(x_0) - \epsilon]$. Thus there exists a subsequence $(s(\bar{x}_n))$ of $(s(x_n))$ with $\lim s(\bar{x}_n) = r_0$. Therefore and since $\lim \bar{x}_n = x_0$, the sequence of non-extenders $(\exp_s(s(\bar{x}_n)\bar{x}_n))$ converges against $\exp_s(r_0, x_0)$. By ***) above $\exp_s(r_0, x_0)$ is a non-extender. However this is a contradiction

because $r_0 < s(x_0)$. Let us treat now case II. Clearly here the geodesic pieces $g_n := \{\exp_s(tx_n) / 0 \leq t \leq s(x_0) + \epsilon\}$ are minimal joins from $\exp_s((s(x_0) + \epsilon)x_n)$ to S . By assertion 2.1a a subsequence $\bar{g}_n := \{\exp_s(t\bar{x}) / 0 \leq t \leq s(x_0) + \epsilon\}$ of the sequence g_n converges against a minimal join

$g_0 = \{\exp_s(tx_0) / 0 \leq t \leq s(x_0) + \epsilon\}$. However this yields a contradiction because $\exp_s(s(x_0)x_0)$ is a non-extender.

We now discuss the case where $s(x_0) = \infty$. We wish

to prove that for every positive real number R_0 exists a natural number n_0 such that $s(x_n) \geq R_0$ for all $n \geq n_0$. The proof

will be indirect. Assume there exists a number K_0 and a subsequence $(s(\tilde{x}_n))$ of the sequence $(s(x_n))$ such that $s(\tilde{x}_n) < K_0$

for all $n \in \mathbb{N}$. Then we have a subsequence $s(\tilde{x}_n)$ of the sequence $s(\tilde{x}_n)$ such that $s(\tilde{x}_n)$ converges against a certain number $h \in [0, K_0]$.

Therefore and since $\lim \tilde{x}_n = x_0$ we get $\exp_s(hx_0) = \lim(\exp_s(s(\tilde{x}_n)\tilde{x}_n))$.

Therefore $\exp_s(hx_0)$ being limit of non-extenders must be a non-extender. Hence we have a contradiction because $h < s(x_0)$. This

completes the proof of theorem 5.3.

The following theorem 5.4 is well known see [19] page 239 and [45] page 98. A proof of theorem 5.4 can be given using considerations, technics and arguments similar to those in the proof of theorem 5.3. Therefore we shall omit a proof of

Theorem 5.4: Let M be a complete, unbordered, C^∞ -smooth Riemannian manifold. Let $T_1 M$ be the unit sphere bundle of M i.e. $T_1 M = \{x \in TM / |x|_{\pi(x)} = 1\}$. Then the function $s: T_1 M \rightarrow (0, \infty]$ $s(x) := \sup \{t / d(\pi(x), \exp_x(tx)) = t\}$ is continuous.

The following theorem 5.5 is well known, too, see [16] p. 241 and [43] p. 131. We shall omit a proof which can be given using theorem 5.4 and considerations similar to those in the proof of theorem 5.3.

Theorem 5.5: Let M be a complete, unbordered, C^∞ -smooth Riemannian manifold. Then the function $\psi: M \rightarrow (0, \infty]$, $\psi(p) := d(p, C_p)$, is continuous.

Let M be a complete, bordered C^∞ -smooth Riemannian manifold with C^∞ -smooth boundary. Then it is not clear whether theorem 5.5 will hold in this case. However the subsequent theorem 5.6 due to Berg, Bishop [13] and Scolozzi [62] implies immediately that in this case at least $d(p, C_p) > 0$ for all $p \in M$.

Theorem 5.6: (Local uniqueness) If M is a C^∞ -smooth Riemannian manifold with C^∞ -smooth boundary, then each point has a neighbourhood in which every pair of points can be joined by a unique shortest path.

Remark: We introduce the function $S_A(\cdot): M \rightarrow [0, \infty]$, $S_A(q) := \sup \{\varepsilon \in \mathbb{R} / q \text{ is } \varepsilon\text{-extender relative to } A\}$,
A any given closed set in a complete n -dimensional manifold.

The function $S_A(\cdot)$ reflects e.g. certain regularity properties of the set A and of ∂M . In order to illustrate this we mention now a few facts without giving detailed proofs because we do not need those facts in the sequel. Using theorem 5.3 and arguments similar to those in the proof of theorem 5.3 it is not difficult to show in case $\partial M = \emptyset$ that the function $S_A(\cdot)$ is continuous if A is a C^2 -smooth submanifold of M or if A is an n -dimensional bordered submanifold with C^2 -smooth boundary. If A is only a C^1 -smooth manifold then $S_A(\cdot)$ need not to be continuous even if $\partial M = \emptyset$. This can e.g. be seen from the properties of Kaufmann's example [37] which are described on page

However if A is a closed C^1 -submanifold of M , $\partial M = \emptyset$, then there exists an open neighbourhood $U(A)$ of A with $S_A(U(A)) \subset (0, \infty]$. This can e.g. be seen using corollary 5.9. It can also be proved more directly by showing that there exists a neighbourhood $U(A)$ of A which does not contain picas relative to A , thus $C_A \cap U(A) = \emptyset$. Namely minimal joins of points in $U(A) \setminus A$ with A are given by geodesic segments which are normal on A . There exist no picas in $U(A)$ because it can be shown that those normal geodesic segments starting in A do not intersect in a small neighbourhood of A . This holds due to the Lipschitz continuity of their initial vectors belonging to the normal bundle of A and can be seen using similar estimations as in the proof of step 1 in theorem 5.2. It is not very hard to see that for a general C^1 -smooth manifold $A \subset M$ there need not to exist an open neighbourhood $U(A)$ of A with $S_A(U(A)) \subset (0, \infty]$. In general we have always an open neighbourhood $U(A)$ with $S_A(U(A)) \subset (0, \infty]$ iff A is an EFP-set, see definition 5.1 and theorem 5.7. Let us consider now the case $\partial M = \emptyset$

and A being a point $p \in M \setminus \partial M$. At least if we make no regularity assumptions for ∂M the function $S_A(\cdot)$ need in general not to be continuous even not in a point $q_0 \in M \setminus \partial M$. Namely the example given in figure 3.2 on p.44 describes a situation where we have a sequence of points (q_n) converging against $q_0 \in M \setminus \partial M$ with $S_A(q_0) = S_p(q_0) = \infty$ but $S_p(q_n) = 0$ for all $n \in \mathbb{N}$. The example in figure 3.2 also shows that the function $p \mapsto \psi(p) := d(p, C_p)$ need not to be continuous in a point $\bar{q} \in M \setminus \partial M$. Namely reflect the cone ∂M at the plane which is orthogonal to the central axis of this cone at it's vertex v . Denote this reflexion of ∂M by CO . CO is boundary of a convex body D . Let \bar{q} be any point in $CO \setminus \{v\}$. Then we have $C_{\bar{q}} = \emptyset$, thus $\psi(v) = \infty$. However, it is obvious from figure 3.2 that we can choose a sequence of points $q_n \in M \setminus D$ with $\lim q_n = \bar{q}$, such that the sequence $(d(q_n, C_{q_n}))$ is bounded from below and above by fixed positive real numbers. Therefore $\psi(\cdot)$ cannot be continuous in \bar{q} .

In [29] Federer investigates in Euclidean space a class of sets which enjoy the so called unique footpoint property. Any set A closed has the unique footpoint property if there is a neighbourhood $U(A)$ of A such that for every point $q \in U(A)$ there exists a unique point $\xi(q)$ of A closest to q . Federer calls sets with this property sets of positive reach. Prior to [29] a similar concept had been studied by Durand [27]. Bangert, Kleinjohann and Walter investigate sets of positive reach in Riemannian manifolds, see [11], [39], [38], [71].

Following Bangert and Kleinjohann we call sets with local unique footpoint property shortly "EFP-sets". All convex sets and all sets with C^2 -smooth boundary belong to this class, see [11]. In Riemannian geometry EFP-sets are important for the investigation

of convex sets see [39] and [72]. Using the concept of cut locus we give here now a simple characterisation of EFP-sets. For this we will first give some definitions.

Definition 5.1: Let M be an unbordered complete C^∞ -smooth Riemannian manifold and let A be a closed subset of M . A is called EFP-set if there exists an open neighbourhood $O(A)$ of A such that for every $q \in O(A)$ there exist a unique point $\xi(q)$ in A with $d(q, \xi(q)) = d(q, A)$. The mapping $\xi: O(A) \longrightarrow A$ is called metric projection onto A .

Theorem 5.7: (Characterisation of EFP-sets.) Let M be a complete, unbordered C^∞ -smooth Riemannian manifold and let A be any closed subset of M . Then A is an EFP-set iff there exists an open neighbourhood U of A such that U does not meet the cut locus C_A of A .

Proof of theorem 5.7: Let U be an open neighbourhood of A with $U \cap C_A = \emptyset$. We show for all $q \in U$ exists a unique point $\xi(q)$ in A with $d(q, \xi(q)) = d(q, A)$. By assertion 2.2 there exists at least one $p \in A$ with $d(q, p) = d(q, A)$. Now assume there is another $\bar{p} \in A$, $\bar{p} \neq p$ such that $d(q, \bar{p}) = d(q, A)$. Then $q \in U$ is obviously a pica relative to A . Thus $q \in U \cap C_A$. This is a contradiction against the condition $U \cap C_A = \emptyset$. Therefore $p = \xi(q)$.

It remains to prove the converse direction in theorem 5.7. For this let A be an EFP-set in M . Then we have an open neighbourhood \tilde{U} of A , such that the metric projection $\xi: \tilde{U} \longrightarrow A$ is well defined. We now show there is a neighbourhood U of A with $C_A \cap U = \emptyset$. By Whiteheads theorem there exists for any $a \in A$ an open

ball $B(a, r_a) := \{x \in M \mid d(x, a) < r_a\}$ such that any two points $x, y \in B(a, r_a)$ have a unique minimal join and this join is contained in $B(a, r_a)$. We define $U := \tilde{U} \cap (\bigcup_{a \in A} B(a, \frac{1}{2}r_a))$ and we claim $C_A \cap U = \emptyset$. Now U is an open neighbourhood of A . Assume $C_A \cap U \neq \emptyset$. Then we have a non-extender $q \in U$. Therefore there exists a pica \bar{q} in U . Now since $\bar{q} \in U$ we have a point $a \in A$ with $\bar{q} \in U \cap B(a, \frac{1}{2}r_a)$. Therefore exists a unique $(*) \xi(q) \in A$ with $d(\bar{q}, A) = d(\bar{q}, \xi(q)) \leq \frac{1}{2}r_a$. Thus we have $\{\bar{q}, \xi(q)\} \subset B(a, r_a)$. Due to the definition of $B(a, r_a)$ there exists only one minimal join $c_{\bar{q}}$ from \bar{q} to $\xi(\bar{q})$. Therefore $(*)$ implies that $c_{\bar{q}}$ is the only minimal join from \bar{q} to A . This a contradiction against the statement that \bar{q} is a pica relative to A . Thus $U \cap C_A = \emptyset$. This complete the proof of theorem 5.7.

Remark: If U is a neighbourhood of A such that $U \cap C_A = \emptyset$ then by the above proof the metric projection $\xi: U \longrightarrow A$ is well defined. However if \tilde{U} is an open neighbourhood of A such that the metric projection is well defined then $\tilde{U} \cap C_A = \emptyset$ does not necessarily hold. Take e.g. as manifold M the cylinder $(S^1 \times \mathbb{R}) \subset \mathbb{R}^3$ and as set $A \subset M$ a straight line parallel to the axis of M . Then M is an open neighbourhood of A such that the metric projection $\xi: M \longrightarrow A$ is well defined. However we have here $C_A \cap M \neq \emptyset$.

In [1] V. Bangert proves a result which can be formulated as follows.

Theorem 5.8 Bangert: Let N, M be two Riemannian manifolds and let $f: M \rightarrow N$ be a $C^{1,1}$ -diffeomorphism. If $A \subset M$ is an EFP-set then also $f(A)$ is an EFP-set in N .

Theorem 5.8 in combination with theorem 5.7 yield immediately

Corollary 5.9: Let M, N be two Riemannian manifold and let $f: M \rightarrow N$ be a $C^{1,1}$ -diffeomorphism. Let A be a closed subset of M such that there is an open set $O \supset A$ with $C_A \cap O = \emptyset$. Then we have an open set $\tilde{O} \supset f(A)$ such that $\tilde{O} \cap C_{f(A)} = \emptyset$.

The last corollary means, if a set avoids locally it's cut locus then this property is invariant under $C^{1,1}$ -diffeomorphism and is independent of the Riemannian structure.

Corollary 5.10: Let A be an EFP-set in an unbordered Riemannian manifold. Then there exists an open set U containing A such that the metric projection $\xi: U \setminus A \rightarrow A$ is locally Lipschitz continuous.

Proof: Choose an open set U such that U contains A and $C_A \cap U = \emptyset$. Then every point in $U \setminus A$ is an extender and therefore a Lipschitz point relative to A . Corollary 5.10 follows by applying theorem 4.5b if we define the function $q \mapsto u(q)$ in theorem 4.5b by $u(q) := d(A, q)$.

Corollary 5.10 is a weak version of a result due to Kleinjohann. Kleinjohann proves in [38] that every point $p \in A$ has a neighbourhood $U(p)$ such that the metric projection $\xi: U(p) \rightarrow A$ is

Lipschitz continuous, if A is an EFP-set. The Euclidean case of Kleinjohann's theorem has been proved by Federer in [29]. For the special class of locally convex EFP-sets, Kleinjohann's theorem was proved in Riemannian manifold by R. Walter in [71].

The subsequent theorem mainly describes for a certain class of manifolds some simple relations between the number of isolated points in the cut locus and topological properties of the related manifold.

Theorem 5.11: Let M be an unbordered n -dimensional complete Riemannian manifold. Let A be a closed bordered n -dimensional C^2 -smooth submanifold of M . We define $N_A := \{q \in M \setminus A \mid d(A, \cdot) \text{ not differentiable in } q\}$ and we denote by J_{N_A} the set of isolated points in N_A . Let $|J_{N_A}|$ be the number of points in J_{N_A} and let $k \in \mathbb{N} \cup \{\infty\}$ be the number of connected components of ∂A . Then the following statements are valid:

- a) We have $k \geq |J_{N_A}|$.
- b) Let us assume now that k is finite. If $|J_{N_A}| \geq k$ then $M \setminus A$ is diffeomorphic to the union of k disjoint open unit discs and ∂A is diffeomorphic to the union of k disjoint unit spheres. Further the cut locus $C_A = J_{N_A}$ and C_A consists of k isolated points.
- c) If $N_A = \emptyset$, then $(M \setminus A) \cup \partial A$ is diffeomorphic to the exterior normal bundle over A .
- d) If A is a single point p and if $|J_{N_p}| \geq 1$ then M is homeomorphic to the unit sphere S^n .

Proof of Theorem 5.11: -Note we shall not always derive a claimed statement in all details, when we think that the claim is an easy consequence of the preceding considerations. On the other hand we can also not avoid being redundant when we describe sometimes overlapping statements in different ways. - If q is an isolated point in the set N_A this means that q is also an isolated point in the cut locus C_A because N_A is a dense subset of C_A by theorem 3.1.

Proof of theorem 5.11a: During the subsequent considerations it will be necessary to prove more than is claimed in theorem 5.11a. However these considerations are also very important for the proof of other parts of theorem 5.11.

We know from above that any point $q \in J_{N_A}$ is an isolated point in the cut locus C_A . We want to show now that for any given isolated point q in C_A exists a connected component R_q of ∂A with the following properties: All geodesics which are normal on R_q meet in q . All these geodesics yield minimal joins to A until they meet in q . For all points $x \in A \setminus R_q$ is $d(x, q) > d(R_q, q)$. We claim further that R_q is diffeomorphic to the $(n-1)$ -dimensional unit sphere S^{n-1} . Precisely we will prove:

(*) "The distance ball $B_{r_q}(q) := \{x \in M / d(x, q) \leq r_q := d(A, q)\}$ is diffeomorphic to the closed n -dimensional unit disc and $\partial B_{r_q}(q) = R_q$ ".

The proof of the preceding statements needs some preparations. For this note A is an n -dimensional bordered submanifold in M . Therefore there exists for any point $b \in \partial A$ exactly one vector $X_b \in T_b M$, $|X_b| = 1$, such that $g(t) := \exp_b(tX_b)$ yields for small

enough positive numbers t a minimal join to A i.e. $d(A, g(t)) = t$, $\| \cdot \|$ being the norm in $T_b M$. The vector X_b is called exterior normal vector relative to A at the point b . We know by theorem 5.3 that the map $s: \partial A \rightarrow [0, \infty]$, $s(b) := \sup\{t \in \mathbb{R} / d(\exp_b(tX_b), A) = t\}$ is continuous. For any $b \in \partial A$ with $s(b) < \infty$ define $\tilde{s}(b) := \exp_b(s(b)X_b)$. Now let c be an arbitrary point in ∂A such that $\tilde{s}(c) = q$ is the above isolated point in C_A . Further let \bar{R}_c be the connected component of ∂A which contains c . We will show now that $s(\bar{R}_c) = d(A, q)$ and $\tilde{s}(\bar{R}_c) = q$, thus $\bar{R}_c \subset S_{r_q}(q) := \{y \in M / d(y, q) = r_q\}$. We use theorem 5.3. The set $R_q := s^{-1}(]0, \infty[) \cap \bar{R}_c$ is open in \bar{R}_c . Since q is both open and closed in C_A , $\tilde{s}^{-1}(\{q\}) \cap \bar{R}_c$ is open and closed in \tilde{R}_q , and hence a full connected component of \tilde{R}_q . On this connected component $s = d(A, q)$. Therefore this component is also closed in \bar{R}_c and hence equals \bar{R}_c .

In order to finish the proof it is sufficient to show the statement (*). For this let $\varepsilon < d(A, q)$ be so small that $B_\varepsilon(q)$ is diffeomorphic to the n -dimensional unit disc. Thus $S_\varepsilon(q) := \{y / d(q, y) = \varepsilon\}$ is diffeomorphic to the $(n-1)$ -dimensional unit sphere S^{n-1} . Now define $\hat{s}: \bar{R}_c \rightarrow M$ by $\hat{s}(p) := \exp_p((r_q - \varepsilon)X_p)$, p any point in \bar{R}_c . Using partly similar arguments as above it is easy to prove that $\hat{s}(\bar{R}_c)$ is open and

closed in $S_\epsilon(q)$. Thus $\hat{S}(\bar{R}_\epsilon) = S_\epsilon(q)$. It is now obvious that $\hat{S} : \bar{R}_\epsilon \longrightarrow S_\epsilon(q)$ is a diffeomorphism, namely recall $(r_q - \epsilon) < s(\bar{R}_\epsilon)$ and remember theorem 5.2. Therefore \bar{R}_ϵ is diffeomorphic to the $(n-1)$ -dimensional unit sphere. Clearly $\partial B_{r_q}(q) = \bar{R}_\epsilon$. Using the diffeomorphism $\hat{S}^{-1} : S_\epsilon(q) \longrightarrow \bar{R}_\epsilon$ it is not difficult to see that $d(q, C_q) > r_q$, thus $\exp_q : B_{r_q}(0) \longrightarrow B_{r_q}(q)$ is a diffeomorphism. Here $B_{r_q}(0) := \{X \in T_q M \mid |X| < r_q\}$, $|\cdot|$ being the norm in $T_q M$. We have shown that $B_{r_q}(q)$ is diffeomorphic to the n -dimensional unit disc. It is obvious that $\overset{\circ}{B}_{r_q}(q) := \{y \in M \mid d(q, y) < r_q\} \subset (M \setminus A)$ and that $\bar{S}(p') \neq q$ for all $p' \in \partial A \setminus \bar{R}_\epsilon$.

We have proved theorem 5.11a because we have shown that for any point $q \in J_{N_A}$ there exists a boundary component which is mapped by \bar{S} on the point q and we have shown that different isolated points in C_A belong different boundary components of ∂A . Therefore we can use the isolated points to describe and distinguish the related boundary components. This means for any point $q \in J_{N_A}$ exists exactly one boundary component R_q with $\bar{S}(R_q) = q$ and if $q, \bar{q} \in J_{N_A}$ with $q \neq \bar{q}$ then $R_q \neq R_{\bar{q}}$. Therefore the number of boundary components of ∂A gives an upper bound for the number of points in J_{N_A} . This is the claim of theorem 5.11a.

Proof of theorem 5.11b: Using theorem 5.11a the statement $|J_{N_A}| \geq k$ implies that $|J_{N_A}| = k$. This means that every boundary component corresponds to a different isolated point in C_A . Now let x be any point in $M \setminus A$. There exists at least one minimal join c_x from x to A which meets ∂A say in a point \bar{x} . By the assumption there exists an isolated point q

in C_A such that \bar{x} is contained in the connected boundary component R_q . Now by the considerations in the proof of theorem 5.11a it is clear that c_x is part of a minimal geodesic between A and q . This geodesic starts in $\bar{x} \in R_q$ and meets the cut locus C_A in q . Therefore x is contained in $\mathring{B}_{r_q}(q) := \{y \in M / d(y, q) < r_q\}$, $M \setminus A \supset \mathring{B}_{r_q}(q)$ being diffeomorphic to an n -dimensional open unit disc. It is obvious from the considerations in the preceding part of the proof that the discs $B_{r_q}(q)$, $q \in J_{N_A}$ are all disjoint i.e. $\mathring{B}_{r_q}(q) \cap \mathring{B}_{r_{\bar{q}}}(\bar{q}) = \emptyset$ if $q \neq \bar{q}$, $q, \bar{q} \in J_{N_A}$. Therefore we have shown that $M \setminus A$ is diffeomorphic to the union of $|J_{N_A}|$ disjoint open unit discs i.e. to $\bigcup_{q \in J_{N_A}} \mathring{B}_{r_q}(q)$. Further clearly $C_A = J_{N_A}$ consists of $|J_{N_A}| = k$ isolated points. Therefore say using (*) above $\partial A = \bigcup_{q \in C_A} R_q = \bigcup_{q \in J_{N_A}} S_{r_q}(q)$ is diffeomorphic to k disjoint unit spheres.

Proof of theorem 5.11c: Using theorem 3.1 the condition $N_A = \emptyset$ implies that the cut locus $C_A = \emptyset$. The claim of theorem 5.11c is now obvious.

Proof of theorem 5.11d: Take a small positive number δ such that $B_\delta(p)$ is diffeomorphic to the closed n -dimensional unit disc. The disc $A := B_\delta(p)$ is a closed bordered n -dimensional submanifold of M . It is easy to see that the cut locus C_A agrees with cut locus C_p . Namely a pica relative to A is obviously also a pica relative to p and vice versa. Therefore C_p and C_A agree on a

dense subset and thus $C_p = C_A$. By the assumption of theorem 5.11d there exists an isolated point q in C_p because N_p is dense in C_p by theorem 3.1. The point q must also be an isolated point of C_A . Now since N_A is a dense subset of C_A , the point q must be an isolated point in N_A . Thus we have

$|J_{N_A}| \geq 1 = k = 1$. Therefore using theorem 5.11b we get that C_A consists of one isolated point q . The considerations in the proof of theorem 5.11b yield further that M is union of two cells namely the discs $B_\delta(p)$, $B_{r_q}(q)$, $r_q = d(A, q)$. These discs are matched together along their common boundaries i.e. $\partial B_\delta(p) = S_\delta(p) = S_{r_q}(q) = \partial B_{r_q}(q)$. It is now easy to construct a homeomorphism between M and the n -dimensional unit sphere.

§ 6 CUT LOCI ON BORDERED SURFACES

As an illustration of the results and methods of §§ 2, 3, 4 we study the cut locus of special bordered surfaces.

Note some of our results will be formulated and proved first for bordered subsurfaces of the Euclidean plane E^2 , because here a technical description of the proof is more comfortable than in the most general case. However, for global results our proofs will only use arguments which transfer literally to the general case where E^2 is replaced by any unbordered, complete, two dimensional Riemannian manifold M with cut locus $C_p = \emptyset$ for all $p \in M$. If we make essentially local considerations, then E^2 can be replaced by any two dimensional Riemannian manifold. Therefore if this is possible the more general results will be merely stated as an obvious consequence after the proof of the special case for subsurfaces of E^2 . Note in order to avoid clumsy descriptions, we will often not distinguish between a parameterized path and its related point set, when we believe that the situation is

clear from the context and a precise description can be supplied easily. Like in the preceding sections $d(\cdot, \cdot)$ will always denote the intrinsic distance function defined in §2.

The following proposition describes bordered surfaces providing examples for our further considerations.

Proposition 6.1': Let S be a closed, connected, topological subsurface of the Euclidean plane $(E^2, \|\cdot\|)$, $\|\cdot\|$ the Euclidean norm. We assume that ∂S contains only locally rectifiable paths; i.e. for any point $p \in \partial S$ we have a Euclidean ball $B_r(p) := \{x \in E^2 \mid \|x-p\| \leq r\}$ such that $B_r(p) \cap \partial S$ is subset of one simple, rectifiable path c contained in ∂S , c homeomorphic to $[0, 1]$. Then (S, d) is a space with an interior metric. This metric space is locally compact and complete. Further (S, d) is homeomorphic to the metric subspace $(S, \|\cdot\|)$ of $(E^2, \|\cdot\|)$, $(S, \|\cdot\|)$ carrying the metric induced by the Euclidean norm.

Proof of proposition 6.1': We show first that any two points in S can be joined by a rectifiable path contained in S . It is obviously sufficient to prove that any point in $S \setminus \partial S$ can be joined with any point in ∂S by a rectifiable path contained in S . Let q be any point in the connected open region $S \setminus \partial S$. The set of points in $S \setminus \partial S$ which can be joined to q by a finite polygon contained in $S \setminus \partial S$ is open and closed in $S \setminus \partial S$ and thus equals $S \setminus \partial S$. Now let p be any point in ∂S . Take a ball $B_r(p)$ such

that $B_r(p) \cap \partial S$ is part of one rectifiable path $c \in \partial S$, c homeomorphic to $[0,1]$. Pick a point \bar{q} in $B_r(p) \cap (S \setminus \partial S)$. Starting in \bar{q} we move now along the Euclidean segment connecting \bar{q} and p until we meet ∂S the first time. Then we move along c until we reach p . Denote the just described path from \bar{q} to p by b . Joining q and \bar{q} by a finite polygon \bar{b} contained in $S \setminus \partial S$, the union $\bar{b} \cup b$ yields a rectifiable path in S from q to p .

Hence any two points in S can be joined by a rectifiable path contained in S . Defining for any two points p, q in S the intrinsic distance $d(p, q) := \inf \{ \text{length } \bar{c} / \bar{c} \text{ a rectifiable path in } S \text{ from } p \text{ to } q \}$ it is not hard to see that (S, d) is a metric space with an interior metric; see also remark 2.1.

If we can show the following implication: (*) "For any sequence (x_n) in S the condition $\lim |x_n - x_0| = 0$ implies $\lim d(x_n, x_0) = 0$, $x_0 \in S$ ". Then we have: (**) " (S, d) is homeomorphic to $(S, | \cdot |)$ ".

For due to $d(x_n, x_0) \geq |x_n - x_0|$ we get from $\lim d(x_n, x_0) = 0$ also $\lim |x_n - x_0| = 0$. Thus using (*) we get (**). Clearly (**) implies the completeness of (S, d) . A Cauchy sequence (x_n) in (S, d) being also a Cauchy sequence in the complete space $(S, | \cdot |)$ is converging in $(S, | \cdot |)$.

Therefore by (**) the sequence (x_n) is converging in (S, d) . This proves the completeness of (S, d) . It is obvious that (**) implies that (S, d) is locally compact.

In order to finish the proof of proposition 6.1 it remains to show (*). For this let (x_n) be any sequence

in S with $\lim |x_n - x_0| = 0$. We consider only the non-trivial case where x_0 is a point in ∂S . From some number n_0 on the sequence (x_n) is contained in a certain ball $B_s(x_0)$, $s > 0$ such that $B_s(x_0) \cap \partial S$ is part of one simple, normalized, rectifiable path $\tilde{c}: [0, \bar{t}] \rightarrow \mathbb{E}^2$, $\tilde{c}[0, \bar{t}] \subset S$, with $\tilde{c}(t_0) = x_0$ for some $t_0 \in]0, \bar{t}[$. For any given $x_n \in B_s(x_0)$ we move now beginning in x_n along the Euclidean segment joining x_n with x_0 until we meet ∂S the first time at some point denoted by $\tilde{c}(t_n)$. Clearly we have $d(x_0, x_n) \leq |t_0 - t_n| + |\tilde{c}(t_n) - x_n| \leq |t_0 - t_n| + |x_n - x_0|$. We are finished if we can show $\lim |t_0 - t_n| = 0$. Assume the contrary. Then we get a subsequence $(t_{n'})$ of (t_n) with $\lim t_{n'} = \bar{t}_0 \neq t_0$, $\bar{t}_0 \in [0, \bar{t}]$. Therefore $\lim |\tilde{c}(t_{n'}) - \tilde{c}(\bar{t}_0)| = 0$ and $\tilde{c}(\bar{t}_0) \neq \tilde{c}(t_0) = x_0$ because $\tilde{c}[0, \bar{t}]$ is simple. This yields a contradiction for $\lim |\tilde{c}(t_{n'}) - x_0| = 0$.

Remark 6.1: It is not difficult to give examples showing that in case ∂S is not locally rectifiable (S, d) need not be locally compact. -It is also easy to give examples showing that under the assumptions of proposition 6.1¹ the metric $d(\cdot, \cdot)$ and the metric induced by the Euclidean norm $|\cdot|$ need not be locally equivalent. Namely define a subsurface S of the Euclidean plane \mathbb{E}^2 as union of the following sets $A := \{(u, v) \in \mathbb{E}^2 / u \leq 0\}$, $B := \{(u, v) \in \mathbb{E}^2 / u > 0, v \in (\mathbb{R}^1 \setminus]0, u^2[)\}$, (u, v) Euclidean coordinates. Choosing in $S := A \cup B$ sequences of points

$$\rho_n := \left(\frac{1}{n}, 0\right), \bar{\rho}_n := \left(\frac{1}{n}, \frac{1}{n^2}\right) \text{ we get } \frac{d(\rho_n, \bar{\rho}_n)}{|\rho_n - \bar{\rho}_n|} > \frac{2\frac{1}{n}}{\frac{1}{n^2}} = 2n,$$

while we have $\lim \rho_n = \lim \bar{\rho}_n = 0$.

The subsequent proposition 6.1 is a trivial generalisation of the preceding proposition 6.1'. The existence of distance realizing paths follows by lemma 2.1.

Proposition 6.1: Let S be a closed, connected topological subsurface of an unbordered, complete, two dimensional C^∞ -smooth Riemannian manifold M . We assume that ∂S contains only locally rectifiable paths. Then (S, d) is a complete space with an interior metric and (S, d) is homeomorphic to the topological subsurface S of M . Therefore (S, d) being locally compact and complete enjoys the Heine-Borel property (c f. [77] p.2) and any two points $p, q \in S$ can be joined by a path in S which has length $d(p, q)$.

Proposition 6.1 is basic for all our further results. It will sometimes be used without any reference. The subsequent lemmata describing properties of the distance function and of cut loci in a special class of surfaces are important for the proof of our main result in this paragraph i.e. theorem 6.2.

Lemma 6.1: Let S be a closed, simply connected topological subsurface of the Euclidean plane E^2 . Assume that ∂S contains only locally rectifiable curves. Then $C_p \cap (S \setminus \partial S) = \emptyset$ for every point $p \in S$, C_p the cut locus in S of the point p . Further the distance function $d(p, \cdot)$ is C^1 -smooth on $S \setminus (\partial S \cup \{p\})$ and has a locally Lipschitz continuous gradient there.

Proof of lemma 6.1: By proposition 6.1 (S, d) is a locally compact space with an interior metric and any two points in S can be joined by a minimal path contained in S . Let p be any point in S . We apply theorem 3.1 and theorem 5.1 for the proof of this lemma. For this we show $(C_p \cap (S \setminus \partial S)) = \emptyset$. Using corollary 3.1 it is sufficient to prove that there is no point $q \in (S \setminus \partial S)$ having (at least) two minimal joins to p with distinct initial vectors in q . Assume the contrary. Then we have in S two normalized minimal paths g_1, g_2 from q to p and $g_1 \setminus \{q\}$, $g_2 \setminus \{q\}$ meet the first time after their start at some point $\bar{p} = g_1(t_0) = g_2(t_0)$, $t_0 > 0$. Moving from q to \bar{p} along g_1 and moving then back to q along g_2 we get a simple closed curve $\alpha \subset S$. "The path α is boundary of a simply connected bounded region R which is contained in S 1)".

1) See remark 6.2.

Now let $g(t)$ be a normalized ray issuing from $q = g(0)$, with initial direction $\dot{g}(0)$, $\dot{g}(0)$ pointing into R and $\dot{g}(0) \neq \dot{g}_1(0)$, $\dot{g}(0) \neq \dot{g}_2(0)$.

"After it's start in q the ray $g(t)$ will stay in $R \subset S$ until $g \setminus \{q\}$ meets α the first time at some point q_m ." ²⁾ The point q_m is contained in $g_1 \cup g_2$. Let us assume $q_m \in g_1$. Moving from q along the ray g to q_m and then from q_m along g_1 to p we get a path \bar{g} from q to p . It is obvious that \bar{g} is shorter than g_1 , a contradiction because $\bar{g} \subset S$. Using the same arguments the assumption $q_m \in g_2$ yields a contradiction as well.

Remark 6.2: The preceding proof makes implicitly use of the Jordan curve theorem, see [26] p.256. We show now that the ray $g(t)$ pointing into the bounded component R of $E^2 \setminus \alpha$ will stay in S until $g \setminus \{q\}$ meets α the first time. For this, we shall prove the following statement: "Let S' be a simply connected subset of E^2 and let $\beta: [0,1] \rightarrow S'$, $\beta(0) = \beta(1) =: q'$ be a simple closed curve in S' . By the Jordan curve theorem we have a bounded component R' of $E^2 \setminus \beta$. We claim that R' is contained in S' ." Assume this claim is not true. Then there exists a point \bar{q} in R' which is not in S' . Define a constant loop γ by $\gamma: [0,1] \rightarrow S'$, $\gamma(t) = q'$. We identify E^2 with the set of complex numbers \mathbb{C} . Using the common notation of complex analysis we obviously get for the

2) See remark 6.2.

$$\underline{\text{index}} \quad j(\gamma, \bar{q}) := \frac{1}{2\pi i} \int_{\gamma} \frac{1}{z - \bar{q}} dz = 0, \quad \text{c.f. [26] page 223, 256}$$

We shall need the following well known fact: (+) For any loop δ in $\mathbb{C} \setminus \{\bar{q}\}$ the index $j(\delta, \bar{q})$ equals $j(\eta, \bar{q})$ for all loops η homotopic to δ in $\mathbb{C} \setminus \{\bar{q}\}$. This statement (+) makes sense even for merely continuous loops which are not necessarily rectifiable, see [26] p.251. Now by the Jordan curve theorem c.f. [26] p.256, we get for an arbitrary point \bar{q} in the bounded component R' of $\mathbb{C} \setminus \{\beta\}$ the index $j(\beta, \bar{q}) \in \{1, -1\}$. Thus $j(\beta, \bar{q}) \in \{1, -1\}$. On the other hand due to the simply connectedness of S' the loop β is homotopic to γ in S' . Therefore and because $\bar{q} \notin S'$ we have β homotopic to γ in $\mathbb{C} \setminus \{\bar{q}\}$. Hence by (+) we get $j(\beta, \bar{q}) = j(\gamma, \bar{q}) = 0$, a contradiction!

Definition and remark 6.3: Let \bar{M} be an unbordered, complete two dimensional, C^∞ -smooth Riemannian manifold. We assume further that \bar{M} is simply connected and has no conjugate points. A Riemannian manifold having all those properties is called a space of type (*).

The following fact is well known: "If an unbordered complete Riemannian manifold M without conjugate points is simply connected then the cut locus C_p is empty for all $p \in M$." Suffice it to say that the absence of conjugate points implies that the exponential $\exp_p: T_p M \rightarrow M$ is a covering map for all $p \in M$. Thus \exp_p is a homeomorphism if M is simply connected. On the other hand if there are no cut points on an unbordered complete Riemannian manifold M then M being diffeomorphic to R^n is simply connected and there are no conjugate points on M .

The following lemma 6.1 is obviously a trivial generalisation of the preceding lemma 6.1'.

Lemma 6.1: Let S be a closed, simply connected sub-surface in a space of type (*). Assume that ∂S contains only locally rectifiable curves. Then $C_p \cap (S \setminus \partial S) = \emptyset$, for any $p \in S$, C_p the cut locus of p in S . Further the distance function $d(p, \cdot)$ is C^1 -smooth on $S \setminus (\partial S \cup \{p\})$ and has a locally Lipschitz continuous gradient there.

Corollary and remark 6.4: The preceding lemma 6.1 showed that in a certain class of simply connected bordered surfaces the cut locus of a point does not meet the interior of the bordered surface. Thus we get by definition 3.4 and theorem 3.1 that the cut locus of any point p in those bordered surfaces is empty; precisely $C_p^I = C_p^{II} = C_p^{IV} = \emptyset$, see § 3 . This means the cut locus C_p in the sense of the definitions 3.4I, 3.4II, 3.4.IV is empty. We have not yet proved that $C_p^{III} = \emptyset$ i.e. we have not yet shown that the cut locus C_p is empty when C_p is described by definition 3.4.III. We shall show this now. Moreover we will prove a stronger result which is of interest per se, i.e. the subsequent theorem 6.1'. Note our proof of theorem 6.1' is constructed such that it can be transferred to more general situations.
c.f.: remark 6.5. Otherwise it would be possible to give a much simpler proof of theorem 6.1'.

Theorem 6.1': Let S be a simply connected subset of E^2 , then any two given points in S can be joined by at most one shortest normalized path contained in S .

Proof of theorem 6.1': The proof will be indirect.
Let $g_1(\cdot), g_2(\cdot): [0, L] \rightarrow S$ be two distinct normalized, shortest paths in S joining the point $p = g_1(0) = g_2(0)$ with the point $q = g_1(L) = g_2(L)$. Now since those paths do not coincide completely there exists $t_0 \in]0, L[$ with $|g_1(t_0) - g_2(t_0)| > 0$, i.e. the Euclidean norm. Taking the connected component of $\{t / |g_1(t) - g_2(t)| > 0, t \in]\}$ which

contains t_0 we get a subintervall $\tilde{J} :=]t_0 - \varepsilon, t_0 + \delta[$, $\varepsilon, \delta > 0$ of J . We have obviously $g_1(t_0 - \varepsilon) = g_2(t_0 - \varepsilon)$, $g_1(t_0 + \delta) = g_2(t_0 + \delta)$ and $g_1(\tilde{J}) \cap g_2(\tilde{J}) = \emptyset$. It is no restriction to assume in the proof from now on that \tilde{J} equals J . Moving along g_1 from p to q and then along g_2 back to p we get a simple closed rectifiable curve α . By remark 6.2 the bounded component R of $E^2 \setminus \alpha$ is contained in S . We know by the Schönflies theorem [53] page 72, that $\tilde{S} := (R \cup \alpha)$ is a topological submanifold of E^2 with boundary α and that \tilde{S} is homeomorphic to the closed unit disc. Therefore \tilde{S} is a closed, simply connected topological subsurface of E^2 with rectifiable boundary curve α . Pick any point $r \in R$. By proposition 6.1 we can join p with r in \tilde{S} by a minimal path $g(t)$, $g(0) = p$. By lemma 6.1 we have $C_p \cap R = \emptyset$, C_p the cut locus of p in \tilde{S} . Therefore by theorem 3.1 the minimal path $g(t)$ (ending up as geodesic segment at $r = g(t_1)$) can be extended as minimal path beyond r until it meets the boundary curve α at some point say $g_1(s) = g(s)$. Moving from r along g back to p , we will obviously stay on a geodesic segment as long as we stay in R until we meet α at a point say $g(t_2)$, $t_2 < t_1$. We claim that $g(t_2) = g_2(t_2)$ with $0 < t_2$. First since $g(\cdot)$, $g_1(\cdot)$, $g_2(\cdot) : [0, t_1] \rightarrow \tilde{S}$ are normalized minimal paths $g(t_2)$ must be contained in $\{g_1(t_2), g_2(t_2)\}$ with $0 \leq t_2 < t_1$. If we would have $g(t_2) \in \{g_1(t) \mid 0 \leq t < t_1\}$ then the (geodesic) segment $g[t_2, s]$ would give in \tilde{S} a shorter join from $g_1(t_2)$ to $g_1(s)$ than the subpath $g_1[t_2, s]$ contradicting the minimal property of g_1 ; note that $g_1[t_2, s]$ cannot coincide

with $g[t_2, s]$ because $g[t_2, s]$ contains the interior point $r \in \tilde{S}$. We assert now that $g(s) = g_1(s) \neq q$ and thus $s < L$, because otherwise we would immediately get a contradiction to the minimal property of g_2 by an argument similar to that one in the preceding sentence.

Next we describe a simple closed curve β in \tilde{S} . We get β by moving from $g_2(t_2)$ along g up to $g(s)$ then following g_1 up to q finally moving along g_2 back to $g_2(t_2)$.

By remark 6.2 the bounded component R_1 of $E^2 \setminus \beta$ is contained in \tilde{S} . Thus again as above $\tilde{S}_1 := R_1 \cup \beta$ is a simply connected, closed topological subsurface of E^2 with rectifiable boundary curve β and \tilde{S}_1 is contained in \tilde{S} . We now claim: (*) "There exists $\varepsilon > 0$ such that $\{g(t) / s \leq t \leq s + \varepsilon\}$ is contained in \tilde{S} , where here as above $g(t)$ denotes for all $t > t_1$ the unique geodesic extension of g beyond r ."

For the proof of (*) we take a small ball

$B_\delta(g(s)) := \{x \in E^2 / |x - g(s)| \leq \delta\}$ such that $\beta \cap B_\delta(g(s))$ is contained in the set $g[t_1, s] \cup g_1[s, \bar{s}]$, \bar{s} some number in $]s, L[$.

Thus there is in $B_\delta(g(s))$ no boundary of \tilde{S}_1 which belongs to g_2 . Assume our claim (*) is not true. Then it is possible to find a number s_1 , $s < s_1 < s + \frac{\delta}{4}$ such that $g(s_1)$ is not contained in \tilde{S}_1 . Pick now a closed half disc H with center $g(s - \frac{\delta}{4})$ and radius $\frac{\delta}{4}$ such that H is contained in $B_\delta(g(s))$ and in \tilde{S}_1 . Next we consider the (geodesic) segments of length $(\frac{\delta}{4} + s_1 - s)$ which start in $g(s - \frac{\delta}{4})$ and meet the interior of H . We can obviously choose among those segments a sequence u_n of segments with endpoints of the

u_n converging against $g(s_1)$. Clearly all u_n are contained in $B_\delta(g(s))$ because

$$\sup\{|x-g(s)| \mid x \in u_n, n \in \mathbb{N}\} \leq (\text{length } u_n) + |g(s) - g(s - \frac{\delta}{4})| \leq \frac{3}{4}\delta < \delta.$$

Now if all u_n would stay in \tilde{S}_1 then $g(s_1)$ would belong to \tilde{S}_1 , because $g(s_1)$ is then a limit of points from the closed set \tilde{S}_1 . Therefore there exists an half open segment $\bar{u}_{n_0} := u_{n_0} \setminus \{g(s - \frac{\delta}{4})\}$ which leaves \tilde{S}_1 . Thus \bar{u}_{n_0} meets the boundary ∂ the first time at a point b which clearly has to be in $g_1([s, \bar{s}])$, say $b = g_1(s_2)$. Moving from p along g up to $g(s - \frac{\delta}{4})$ moving then along u_{n_0} up to b we get in \tilde{S} a path from p to $g_1(s_2)$. The length of this path is obviously shorter than s_2 , remember g is a minimal join between p and $g(s) = g_1(s)$. Thus we have got a contradiction to the minimal property of g_1 . This proves (*). Defining $\epsilon_m := \max\{\epsilon \mid (g[s, s+\epsilon]) \subset \tilde{S}_1\}$ we get by (*) that $\epsilon_m > 0$. Now $g(s+\epsilon_m)$ is obviously a point in the boundary ∂ of \tilde{S}_1 . Thus $g(s+\epsilon_m)$ is a point in $g_1[s, L[\cup g_2]t_2, L]$ because $g(s+\epsilon_m) \notin g[t_2, s]$ and since $g(t_2) = g_2(t_2)$. If $g(s+\epsilon_m)$ is in $g_2]t_2, L]$ then we get a contradiction to the minimal property of g_2 . Therefore $g(s+\epsilon_m)$ is in $g_1[s, L[$ and this implies obviously that $g_1[s, s+\epsilon_m]$ coincides with the segment $g[s, s+\epsilon_m]$. Now we define a simple closed curve β_2 in \tilde{S}_1 . We describe β_2 as follows. We move from $g(t_2)$ along g to $g(s+\epsilon_m)$ then along g_1 to $g_1(L)$, recall $L > (s+\epsilon_m)$. Finally we move from $g_1(L)$

along g_2 back to $g(t_2)$. Let R_2 be the bounded component of $E^2 \setminus \beta_2$. As shown above we get a simply connected, closed subsurface $S_2 := (\beta_2 \cup R_2) \subset \tilde{S}_1$. Due to the definition of ϵ_m we know that for all $\eta > 0$ the segment $g[s, s + \epsilon_m + \eta]$ contains points outside of S_2 . This yields a contradiction to the minimal property of g_1 by the same argument we used above to prove (*).

The preceding proof used only that S is a simply connected subset of a two dimensional complete unbordered Riemannian manifold M , where M has not cut points. Therefore we get the following generalisation of theorem 6.1'.

Theorem 6.1: Let S be a simply connected subset in a space M of type (*), c.f.: definition 6.3. Then any two points of S can be joined by at most one shortest normalized path contained in S .

Remark 6.5: For the case that M equals E^2 one could have given much shorter proofs for theorem 6.1, using the fact that in E^2 all distance balls are strongly geodesically convex. Those proofs do not work in spaces of type (*). Namely there exist complete twodimensional Riemannian manifolds without cut points, which have an area of positive curvature and posses focal points, see [34] p.192. In those manifolds geodesic balls are not necessarily convex. -We wish to point out that our proofs of lemma 6.1 and theorem 6.1 seem to work also when the space of

type (*) there is replaced by a two dimensional metrically complete Finsler manifold M , M without cut points and $\partial M = \emptyset$.

The preceding lemma 6.1 and theorem 6.1 showed that in a certain class of simply connected bordered surfaces the cut locus of a point does not meet the interior of the bordered surface, moreover this cut locus is empty. We shall treat now the converse problem. We will prove in lemma 6.2, that for a complete bordered surface which is not simply connected the cut locus of any point will always meet the interior of the surface. For this we need first the following technical proposition.

Proposition 6.2: Let \bar{M} be a two-dimensional, unbordered complete, C^∞ -smooth Riemannian manifold. Let M be a closed, connected, topological subsurface of \bar{M} . We assume that ∂M contains only locally rectifiable paths. Let p_1, p_2 be two distinct points in M and $c: [0, 2l] \rightarrow M$, a normalized, minimal join from p_1 to p_2 . Then we can find a sequence of points $q_n \in M \setminus \partial M$ with $\lim q_n = q := c(1)$ and $l_n := d(p_1, q_n) = d(p_2, q_n)$, and with the following property: if we define normalized paths $c_n := [0, 2l_n] \rightarrow M$ such that $c_n [0, l_n]$ is a minimal join from p_1 to q_n and $c_n [l_n, 2l_n]$ a minimal join from q_n to p_2 then the paths $c_n [0, 2l_n]$ are simple.

Proof of proposition 6.2: In case $q \in (M \setminus \partial M)$ the proposition 6.2 is trivial, because we can choose in this case $q_n = q$ and the claim of the proposition is obviously true. Therefore let us assume that $q \in \partial M$. The proposition 6.1 showed (+): "The topology induced by the intrinsic distance $d(\cdot, \cdot)$ on M agrees with the submanifold topology. Therefore there exists an open neighbourhood U of q such that $B_{\frac{1}{2}}(q) := \{x \in M / d(x, q) \leq \frac{1}{2}\} \supset U$ and U is homeomorphic to the open half disc $\mathring{H}_1(0) := \{(v, w) \in \mathbb{R}^2 / v^2 + w^2 < 1, w \geq 0\}$, (v, w) being Euclidean coordinates in \mathbb{R}^2 . Identifying points of U with the corresponding points in $\mathring{H}_1(0)$ we have $\partial M \cap U = \{(v, w) \in \mathring{H}_1(0) / w = 0\}$. We define two real numbers $f := \min\{s > 0 / |c(1-s)| = \frac{1}{2}\}$, $h := \min\{s > 0 / |c(1+s)| = \frac{1}{2}\}$. Now $c: [1-f, 1+h] \rightarrow H_1(0)$ being part of a minimal join is a simple path. We get a simple closed path $g(t)$, $g: [1-f, 1+h+1] \rightarrow M$ by moving for $t \in [1-f, 1+h]$ from $c(1-f)$ to $c(1+h)$ along $c[1-f, 1+h] =: F$ and moving for $t \in [1+h, 1+h+1]$ on the subarc E of $\{(v, w) \in \mathbb{R}^2 / v^2 + w^2 = \frac{1}{4}, w \geq 0\}$ which joins $c(1+h)$ with $c(1-f)$. Now $g[1-f, 1+h+1] =: G$ being a Jordan curve is boundary of a simply connected subset $G \subset H_1(0) \subset B_{\frac{1}{2}}(q)$ and G is homeomorphic to the closed twodimensional unit disc D . In the preceding statement we made use of the Schönflies-theorem and the Jordan curve theorem. It is clear that by it's description G is defined uniquely, see remark 6.2. Defining $\delta := d(E, q)$ it is obvious that $\delta > 0$. Using that

G is homeomorphic to the closed unit disc D we identify points in G with the corresponding points in D . Therefore E and F are two non trivial subarcs of the unit circle. Now since $q=c(1)$ is an interior point of F it is not difficult to see that for any $n \in \mathbb{N}$ with $\frac{1}{n} < \frac{\delta}{40}$ exist two positive numbers $\alpha_n, \beta_n < \frac{\delta}{40}$ and a path $b_n: [0, 1] \rightarrow G$ with $b_n(0)=c(1-\alpha_n)$, $b_n(1)=c(1+\beta_n)$ such that $b_n[0, 1] \subset G \setminus g$ and $b_n[0, 1] \subset B_{\frac{1}{n}}(q)$. Note by (+) we have γ_n such that $H_{\gamma_n}(0) := \{(v, w) \in H_1(0) \mid v^2 + w^2 \leq \gamma_n\} \subset B_{\frac{1}{n}}(q)$. Using the homeomorphy of D with G we can arrange that $b_n[0, 1]$ is contained in $G \cap H_{\gamma_n}(0)$. Here the path $b_n[0, 1]$ can be got from a sequence of secants T_n in D , the end points of which converge against $q \in T_n$. We define a function $\varphi_n(s) := d(p_1, b_n(s)) - d(p_2, b_n(s))$, $0 \leq s \leq 1$. Using that $c[0, 21]$ is a minimal join we have $\varphi_n(0) = -2\alpha_n < 0$, $\varphi_n(1) = 2\beta_n > 0$. Therefore exists $s_n \in (0, 1)$ with $\varphi_n(s_n) = 0$. Thus we have $q_n := b_n(s_n) \in ((G \setminus g) \cap B_{\frac{1}{n}}(q))$ with $d(p_1, q_n) = d(p_2, q_n) =: l_n$.

1) We assume further condition (K): " The number $\frac{1}{n}$ is so small that for all points $p \in \{x \in \bar{M} \mid \bar{d}(q, x) \leq \frac{\delta}{n}\} = B$, $C_p \cap B = \emptyset$, $d(\cdot, \cdot)$ being the distance in the Riemannian manifold \bar{M} , and \bar{C}_p being the cut locus of the point p in the Riemannian manifold \bar{M} .

Using proposition 6.1 and lemma 2.1 we know that any two points in M can be joined by a distance realizing path in M . Therefore we can define a (normalized) path $c_n: [0, 2l_n] \rightarrow M$ such that $c_n[0, l_n]$ is a minimal join from p_1 to q_n and $c_n[l_n, 2l_n]$ is a minimal join from q_n to p_2 . In order to complete the proof of proposition 6.2 it remains to show that $c_n: [0, 2l_n] \rightarrow M$ is a simple path. Clearly $c_n[0, l_n]$ and $c_n[l_n, 2l_n]$ being minimal joins are both simple. Therefore if $c_n: [0, 2l_n] \rightarrow M$ is not simple, there must exist positive numbers $w_n, \hat{w}_n \in]0, l_n[$ such that $c_n(l_n - w_n) = c_n(l_n + \hat{w}_n)$ thus $d(c_n(l_n), c_n(l_n - w_n)) = d(c_n(l_n), c_n(l_n + \hat{w}_n))$. This yields $w_n = \hat{w}_n$ because $c_n[0, l_n], c_n[l_n, 2l_n]$ are both normalized, minimal joins. We define $\bar{w}_n := \max\{w > 0 / c_n(l_n - w) = c_n(l_n + w)\}^1$. The union $c_n[0, l_n - \bar{w}_n] \cup c_n[l_n + \bar{w}_n, 2l_n]$ being a join from p_1 to p_2 is a path of length $(2l_n - 2\bar{w}_n) \stackrel{(a)}{\geq} d(p_1, p_2) = 2l$. On the other hand since $c_n[0, l_n]$ is a normalized minimal join from p_1 to q_n we get $l_n \leq d(p_1, q) + d(q, q_n) \leq l + d(q, q_n) \leq l + \frac{1}{n}$. Thus we have $2l_n \leq 2l + 2(\frac{1}{n})$. Therefore using inequality (a) we get $2l + 2\bar{w}_n \leq 2l_n \leq 2l + 2(\frac{1}{n})$ thus $\bar{w}_n \leq \frac{1}{n}$. This yields defining $\bar{q}_n := c_n(l_n - \bar{w}_n)$ that $d(\bar{q}_n, q) \leq d(\bar{q}_n, q_n) + d(q_n, q) \leq d(\bar{q}_n, c_n(l_n)) + d(q_n, q) \leq \frac{2}{n}$. Clearly \bar{q}_n is a bifurcation point of minimal joins. Minimal joins being geodesic segments in $G \setminus \partial G$ do not have bifurcation points in $G \setminus \partial G$. Therefore we can assume $\bar{q}_n \notin G \setminus \partial G$. Defining $w'_n := \min\{w / c_n(l_n + w) \in \partial G\}$, we get $0 < w'_n \leq \bar{w}_n \leq \frac{1}{n}$ and $\bar{q}'_n = c_n(l_n - w'_n) = c_n(l_n + w'_n) \in \text{EvF} = \partial G$.

1) Note that $c_n(l_n - w) = c_n(l_n + w)$ for all $w \in [0, \bar{w}_n[$ because we have here uniqueness for minimal joins. This uniqueness holds by theorem 6.1 due to condition (K) because we shall see below that $0 \leq \bar{w}_n \leq \frac{1}{n}$.

This yields $q_n \in F$ because $d(q_n, q) < \frac{2}{n} < \delta = d(E, q)$. Therefore $q_n \in c[0, 21]$. Now we have $d(p_1, q_n) = d(q_n, p_2) = 1_n - w_n$ by the definition of c_n and w_n . However q is the only point in $c[0, 21]$ with $d(p_1, q) = d(q, p_2)$. Thus we get $q = q_n = c_n(1_n - w_n)$ and $d(p_1, q_n) = d(q_n, p_2) = 1_n - w_n$. Therefore we have two minimal joins a_1, a_2 built by the following unions of segments $a_1 := c[1 - \frac{1}{n}, 1] \cup \tilde{g}$, $a_2 := \tilde{g} \cup c[1, 1 + \frac{1}{n}]$, $\tilde{g} := c_n[1_n, 1_n + w_n]$. It is obvious by the definition of w_n that \tilde{g} is a minimal geodesic joining q_n with $q = q_n$. Extending the geodesic segment \tilde{g} by length $\frac{1}{n}$ beyond q we get due to condition (K) on p a minimal geodesic segment \bar{g} which starts in q_n . Now since $c[0, 21]$ is simple it is clear that not all points of $c[1 - \frac{1}{n}, 1[\cup c[1, 1 + \frac{1}{n}]$ are contained in \bar{g} because $\tilde{g} \cap c[1 - \frac{1}{n}, 1[= \emptyset$, $\tilde{g} \cap c[1, 1 + \frac{1}{n}] = \emptyset$, for a_1 and a_2 being minimal joins are simple. Let $q' \in c[1 - \frac{1}{n}, 1[\cup c[1, 1 + \frac{1}{n}]$ be such a point which is not contained in \bar{g} . There exists a minimal geodesic segment g_1 in \bar{M} which joins $q_n \in G \cap \partial G$ with q' and we have $(g_1 \setminus \{q_n\}) \cap (\bar{g} \setminus \{q_n\}) = \emptyset$. Namely, since $(\text{length } g_1) \leq d(q', q) + d(q, q_n) \leq \frac{2}{n} < \text{length } \bar{g}$ the assumption $(g_1 \setminus \{q_n\}) \cap (\bar{g} \setminus \{q_n\}) \neq \emptyset$ would imply $g_1 \subset \bar{g}$ and thus $q' \in \bar{g}$, a contradiction. Now g_1 starting in q_n meets $\partial G = E \cup F$ the first time in a point $\bar{q} \in c[1 - \frac{3}{n}, 1] \cup c[1, 1 + \frac{3}{n}]$ because $d(\bar{q}, q) \leq \frac{3}{n}$ for $d(\bar{q}, q) \leq d(\bar{q}, q_n) + d(q_n, q) \leq (\text{length } g_1) + \frac{1}{n} \leq \frac{2}{n} + \frac{1}{n}$. Let \bar{g}_1 be the partial segment of g_1 which starts in q_n and ends in \bar{q} where say $\bar{q} = c(1 - \theta)$, $0 < \theta \leq \frac{3}{n}$. Clearly

$\bar{g}_1 \subset M$ is a non trivial, minimal geodesic segment from q_n to \bar{q} . The union $c[1-\beta, 1] \cup \tilde{g}$ being contained in a_1 is also a minimal join from q_n to \bar{q} . Therefore we get $(c[1-\beta, 1] \cup \tilde{g}) = \bar{g}_1$ for due to condition (K), g_1 does not meet the cut locus of q_n in \bar{M} . Consequently $g \subset \bar{g}_1$. This yields a contradiction because $(\tilde{g} \setminus \{q_n\}) \cap (g_1 \setminus \{q_n\}) = \emptyset$ and $\bar{g}_1 \subset g_1$. The case where $\bar{q} \in c]1, 1 + \frac{3}{n}]$ can be treated in the same way. This proves proposition 6.2.

Remark 6.6: The second part of the preceding proof (showing that $c_n[0, 2l_n]$ is simple) proves that under certain conditions minimal paths in a bordered manifold do not bifurcate. Using considerations similar to those in the preceding proof it is now easy to show the following statement: "Let p_1, p_2, p_3 be three distinct points in a bordered surface as assumed in proposition 6.2. Now if there exists $q \in M$ such that $d(p_1, q) + d(q, p_3) = d(p_1, p_3)$ and $d(p_2, q) + d(q, p_3) = d(p_2, p_3)$ then $d(p_2, q) + d(q, p_1) > d(p_2, p_1)$."

Lemma 6.2: Let \bar{M} be an unbordered, complete two dimensional, C^∞ -smooth Riemannian manifold. Let M be a closed, connected topological subsurface of \bar{M} . We assume that ∂M contains only locally rectifiable paths. If M is not simply connected then we have $C_p \cap (M \setminus \partial M) \neq \emptyset$ for all points $p \in M$, C_p being the cut locus in M relative to the point p .

Proof of lemma 6.2: We know by proposition 6.1 that M is a complete, locally simply connected space with an interior metric $d(,)$, see also § 2 .

Therefore and because M is not simply connected we know by the assertion in [7] page 11, that for any base point $p \in M$ there exists a shortest non contractible (normalized) loop $c(t)$ in M with $c(0)=c(2l)=p$. Let (\tilde{M}, \tilde{d}) be the universal covering space of M . The space (\tilde{M}, \tilde{d}) is a complete space with an interior metric $\tilde{d}(.,.)$. The covering mapping $\pi: (\tilde{M}, \tilde{d}) \rightarrow (M, d)$ is a local isometry . Let \tilde{p}_1 be any point in the fiber $\pi^{-1}(p)$. Now we lift the path $c(t)$ to \tilde{M} . We start the lift in \tilde{p}_1 and denote the lifted path by $\tilde{c}(t)$, thus $\tilde{c}(0) = \tilde{p}_1$. The lifted path ends up in a point $\tilde{c}(2l) =: \tilde{p}_2 \in \pi^{-1}(p)$. Clearly $\tilde{p}_1 \neq \tilde{p}_2$ for $c(t)$ is not contractible. The (normalized) path $\tilde{c}(t)$, $0 \leq t \leq 2l$ is a minimal join from \tilde{p}_1 to \tilde{p}_2 because $(\pi \circ c)(t)$ is a shortest non contractible loop with base point p . Consequently $\text{length}(\tilde{c}) = 2l = \tilde{d}(\tilde{p}_1, \tilde{p}_2)$. We define $\tilde{q} := \tilde{c}(l)$ and have $\tilde{d}(\tilde{p}_1, \tilde{q}) = \tilde{d}(\tilde{q}, \tilde{p}_2) = l$. Now if $\tilde{q} \in \tilde{M} \setminus \partial \tilde{M}$ then it is easy to see that $\pi(\tilde{q}) =: q \in C_p \cap (M \setminus \partial M)$. Namely in this case $c[0, l]$ and $c[l, 2l]$ yield two minimal joins from q to p with distinct initial vectors at q . This can be seen as follows. Assume $d(c(l), p) =: \bar{l} < l$. Then we have a normalized path $\bar{c}: [0, \bar{l}] \rightarrow M$ from $q=c(l)=\bar{c}(0)$ to the point $p=\bar{c}(\bar{l})$. We define a new path $\hat{c}: [0, l+\bar{l}] \rightarrow M$ by $\hat{c}(t) := c(t)$ if $0 \leq t \leq l$, and $\hat{c}(t) := \bar{c}(t-l)$

if $1 \leq t \leq 1 + \bar{1}$. Now lift the path \hat{c} to \tilde{M} . We start the lift in \tilde{p}_1 and denote the lifted path by $\tilde{c}(t)$. Clearly $\tilde{c}(1) = \tilde{q}$ and $\tilde{c}(1 + \bar{1}) \in \pi^{-1}(p)$. Thus we have two possibilities. First $\tilde{c}(1 + \bar{1}) = \tilde{p}_1$, however this yields a contradiction because then $\tilde{c}[1, 1 + \bar{1}]$ being a join from \tilde{q} to \tilde{p}_1 has length $\bar{1} < 1 = d(\tilde{p}_1, \tilde{q})$. If in the second case $\tilde{c}(1 + \bar{1}) \in \pi^{-1}(p) \setminus \{\tilde{p}_1\}$ then $(\pi \circ \tilde{c})(t)$, $0 \leq t \leq 1 + \bar{1}$ yields a non contractible loop with base point p and length smaller than $2l$, a contradiction. Therefore we can assume that $\tilde{q} \in \partial\tilde{M}$. By proposition 6.2 we have in an arbitrary small neighbourhood of \tilde{q} a sequence of points $\tilde{q}_n \in \tilde{M} \setminus \partial\tilde{M}$ with $\lim \tilde{q}_n = \tilde{q}$ and $\tilde{d}(\tilde{p}_1, \tilde{q}_n) = \tilde{d}(\tilde{q}_n, \tilde{p}_2) = l_n$. Further we have (normalized) simple paths $\tilde{c}_n: [0, 2l_n] \rightarrow \tilde{M}$ with $\tilde{c}_n(0) = \tilde{p}_1$, $\tilde{c}_n(2l_n) = \tilde{p}_2$ and $\tilde{c}_n(l_n) = \tilde{q}_n$. If now $d(p, \pi(\tilde{q}_n)) = l_n$ for some $n \in \mathbb{N}$, then it is obvious that $q_n := \pi(\tilde{q}_n)$ is a point in $C_p \cap (M \setminus \partial M)$. Therefore let us consider the possibility that $d(p, q_n) := \bar{l}_n < l_n$ for all $n \in \mathbb{N}$. Then we have for every $n \in \mathbb{N}$ a (normalized) minimal join $\bar{c}_n(t)$, $0 \leq t \leq \bar{l}_n$ with $\bar{c}_n(0) = q_n$, $\bar{c}_n(\bar{l}_n) = p$. We define now in \tilde{M} a (normalized) paths $\hat{c}_n(t)$, $0 \leq t \leq l_n + \bar{l}_n$ by $\hat{c}_n(t) := \tilde{c}_n(t)$ if $0 \leq t \leq l_n$ and $\hat{c}_n(t) := \tilde{c}_n(t - l_n)$ if $0 \leq t \leq l_n + \bar{l}_n$, $\tilde{c}_n(t)$ being the lifted path of $\bar{c}_n(t)$ where the lift starts in \tilde{q}_n . Clearly $\hat{p}_n := \hat{c}_n(l_n + \bar{l}_n)$ is a point in $\pi^{-1}(p)$. Now we have $\hat{p}_n \neq \tilde{p}_1$ because otherwise $\hat{c}_n[1_n, 1_n + \bar{l}_n]$ would give a join from \tilde{q}_n to \tilde{p}_1 of length $\bar{l}_n < l_n = \tilde{d}(\tilde{p}_1, \tilde{q}_n)$, a contradiction. Further $\hat{p}_n \neq \tilde{p}_2$ because $\bar{l}_n < l_n = \tilde{d}(\tilde{q}_n, \tilde{p}_2)$. Consequently $\hat{p}_n \in \pi^{-1}(p) \setminus \{\tilde{p}_1, \tilde{p}_2\}$. It is obvious that

there exists a number $r > 0$ such that

$(\{\hat{p}_n \mid n \in \mathbb{N}\} \setminus \{\tilde{p}_1, \tilde{p}_2\}) \subset B_r(\tilde{q})$, with

$B_r(\tilde{q}) := \{x \in \tilde{M} \mid d(\tilde{q}, x) \leq r\}$. Now $B_r(\tilde{q}) \cap (\pi^{-1}(p) \setminus \{\tilde{p}_1, \tilde{p}_2\})$

is compact. Moreover it is a finite set. Recall (\tilde{M}, \tilde{d})

has the Heine-Borel property and $\pi: (\tilde{M}, \tilde{d}) \rightarrow (M, d)$ is

a local isometry. Thus we have an equally denoted con-

stant subsequence of (\hat{p}_n) with $\hat{p}_n := \tilde{p}_3 \in \pi^{-1}(p) \setminus \{\tilde{p}_1, \tilde{p}_2\}$

for all $n \in \mathbb{N}$. Now $l_n > \bar{l}_n \geq \tilde{d}(\tilde{q}_n, \hat{p}_n) = \tilde{d}(\tilde{q}_n, \tilde{p}_3)$. Therefore

and because $\lim l_n = \tilde{d}(\tilde{p}_1, \tilde{q}) = 1$ we have $\lim \tilde{d}(\tilde{q}_n, \tilde{p}_3) =$

$\tilde{d}(\lim \tilde{q}_n, \tilde{p}_3) = \tilde{d}(\tilde{q}, \tilde{p}_3) \leq 1$. Hence $\tilde{d}(\tilde{p}_1, \tilde{p}_3) \leq \tilde{d}(\tilde{p}_1, \tilde{q}) +$

$\tilde{d}(\tilde{q}, \tilde{p}_3) \leq 2l$. Now the length of a shortest non contrac-

tible loop with basepoint p is $2l$. Thus $\tilde{d}(\tilde{p}_1, \tilde{p}_3) \geq 2l$

because $\pi(\tilde{p}_1) = \pi(\tilde{p}_3) = p$ and $\tilde{p}_1 \neq \tilde{p}_3$. Consequently

$\tilde{d}(\tilde{p}_1, \tilde{q}) + \tilde{d}(\tilde{q}, \tilde{p}_3) = \tilde{d}(\tilde{p}_1, \tilde{p}_3) = 2l$.

Using the same arguments we also get

$\tilde{d}(\tilde{p}_2, \tilde{q}) + \tilde{d}(\tilde{q}, \tilde{p}_3) = \tilde{d}(\tilde{p}_2, \tilde{p}_3) = 2l$.

On the other hand we also have

$\tilde{d}(\tilde{p}_1, \tilde{q}) + \tilde{d}(\tilde{q}, \tilde{p}_2) = \tilde{d}(\tilde{p}_1, \tilde{p}_2) = 2l$.

However this is not possible by remark 6.6. This proves lemma 6.2.

Remark 6.7: It is easily seen that the considerations in the preceding proof (with trivial modifications) show more than we claimed. Under the assumptions of lemma 6.1 the following holds: "Let p be any point in M . Then we have a point $q \in M$ and sequence $q_n \in C_p \cap (M \setminus \partial M)$ with $\lim q_n = q$."

Combining lemma 6.1 and lemma 6.2 with theorem 6.1 we get now immediately the main result of this paragraph:

Theorem 6.2: Let M be a closed topological subsurface of a two dimensional simply connected, complete Riemannian manifold \bar{M} , \bar{M} without conjugate points and assume that ∂M contains only locally rectifiable curves and $\partial \bar{M} = \emptyset$. Then the following statements are equivalent:

- a) The subsurface M is simply connected.
- b) There exists a point p in M with $C_p \setminus \partial M = \emptyset$, C_p the cut locus in S of the point p .
- c) $C_p = \emptyset$ for all points $p \in M$.
- d) There exists a point p in M such that the distance function $d(p, \cdot)$ is C^1 -smooth on $S \setminus (\partial M \cup \{p\})$.
- e) For all points $p \in M$ $d(p, \cdot)$ is C^1 -smooth on $S \setminus (\partial M \cup \{p\})$ and has a locally Lipschitz continuous gradient there.
- f) Any two points of M can be joined by exactly one shortest normalized path contained in M .
- g) Any two points of $M \setminus \partial M$ can be joined by exactly one shortest normalized path contained in M .

Remark 6.8: In the rest of this paragraph we shall employ the Jordan curve theorem and the Schönflies theorem so many times that we will not always refer to those results. However sometimes we will use the notation (J) as a reference to those theorems.

If a closed bordered subsurface of a space of type (*)¹⁾ has locally rectifiable boundary curves then we call this subsurface a space of type (**). We proceed now with an investigation of the cut locus of two points in a simply connected space of type (**). For this purpose we introduce also the notation of an "equidistancial" set of two points.

Definition 6.4: Let p, q be any two points in a metric space (M, d) . The set $A(p, q) := \{x \in M / d(p, x) = d(x, q)\}$ is called equidistancial set of the points p and q .

The following definition will be helpful to simplify our descriptions.

Definition 6.5: A generalized pica relative to some closed set K is a point y with the following property. There exist at least two normalized minimal joins $g_1 [0, d(y, K)]$, $g_2 [0, d(y, K)]$ going from y to K and there is a number $\varepsilon > 0$ such that $g_1 [0, \varepsilon] \cap g_2 [0, \varepsilon] = \{y\}$.

The subsequent lemma 6.3 is basic for our further considerations.

1) See definition and remark 6.3!

Lemma 6.3: Let p, q be any two distinct points in a simply connected space (M, d) of type (**), M being subsurface of a space (\hat{M}, \hat{d}) of type (*). Let $(y_n) \in M$ be any convergent sequence of generalized picas relative to $\{p, q\}$, say $\lim y_n = y_0$. Then $y_0 \in A(p, q)$ is a generalized pica relative to $\{p, q\}$. Further, if $g_p(t), g_q(t)$ are arbitrary (normalized) minimal joins from y_0 to p and from y_0 to q respectively then $g_p \cap g_q = \{y_0\}$.

Proof of lemma 6.3: Every point y_n is a generalized pica relative to $\{p, q\}$. Therefore there exist two normalized minimal joins g_{1n}, g_{2n} from y_n to $\{p, q\}$ and we have a number $\varepsilon_n > 0$ such that $g_{1n} [0, \varepsilon_n] \cap g_{2n} [0, \varepsilon_n] \stackrel{6.1}{=} \{y_n\}$. Clearly $g_{1n} (d(y_n, \{p, q\})) \in \{p, q\}$. Say $g_{1n} (d(y_n, \{p, q\})) = p$. Then we have $g_{2n} (d(y_n, \{p, q\})) = q$, because otherwise the uniqueness of minimal joins in (M, d) is violated; recall theorem 6.1 is valid for a simply connected space of type (**). Further also due to the "uniqueness of minimal joins" we have $g_{1n} [0, d(y_n, \{p, q\})] \cap g_{2n} [0, d(y_n, \{p, q\})] = \{y_n\}$ because of (6.1). Therefore $y_n \in A(p, q)$ and we have for every point y_n normalized minimal joins g_{pn}, g_{qn} from y_n to p, q respectively with $(g_{pn} \cap g_{qn}) = \{y_n\}$. Clearly $y_0 \in A(p, q)$ because $A(p, q)$ is closed. Thus we have two normalized minimal joins $g_{p0} [0, d(y_0, p)], g_{q0} [0, d(y_0, q)]$ from y_0 to p, q respectively with $d(y_0, p) = d(y_0, q) = d(y_0, \{p, q\}) = 1$. The proof of the lemma is finished if we can show that $g_{p0} \cap g_{q0} = \{y_0\}$. Assume the contrary. Then the number $t_0 := \max \{t / g_{p0}(t) = g_{q0}(t)\}$ is positive. Clearly we have

$g_{p_0} [0, t_0] = g_{q_0} [0, t_0]$ because of theorem 6.1.

Therefore $g_{p_0}(t_0) = g_{q_0}(t_0) =: x_0$ being a bifurcation point of minimal joins is a point in ∂M . Let (y_n) be a subsequence of (y_n) such that g_{p_n} converges against a minimal join \bar{g}_{p_0} from y_0 to p . Then $\bar{g}_{p_0} = g_{p_0}$ by theorem 6.1.

Let (y_n^*) be a subsequence of (y_n) such that $(g_{q_n^*})$ converges against a minimal join from y_0 to q . Then as above $(g_{q_n^*})$ must converge against g_{q_0} . Clearly the sequence $(g_{p_n^*})$ being a subsequence of (g_{p_n}) must still converge against g_{p_0} . In order to simplify the notation we will denote $(g_{p_n^*})$, $(g_{q_n^*})$ by (g_{p_n}) , (g_{q_n}) respectively. The rest of the proof is performed now in several steps. The hardest

step is to prove the following statement (T): "There exists a natural number \bar{n} such that for all $n \geq \bar{n}$ at least one of the two sets $g_{q_n} \cap g_{p_0}$, $g_{p_n} \cap g_{q_0}$ is not empty."

We shall prove T later. Using T we finish now the proof of lemma 6.3. We know by (T) that there exists a number \bar{n} such that say $(g_{q_{\bar{n}}} \cap g_{p_0}) \neq \emptyset$. Thus let $x_{\bar{n}} = g_{p_0}(\bar{t}) = g_{q_{\bar{n}}}(t_{\bar{n}})$. Now in case $\bar{t} \leq t_0$ we have $d(x_{\bar{n}}, p) = d(x_{\bar{n}}, q)$. Therefore if $l_{\bar{n}} := d(y_{\bar{n}}, q) = d(\bar{y}_{\bar{n}}, p)$

$$\text{then } d(y_{\bar{n}}, x_{\bar{n}}) + d(x_{\bar{n}}, p) = d(y_{\bar{n}}, x_{\bar{n}}) + d(x_{\bar{n}}, q) = \\ (\text{length } g_{q_{\bar{n}}} [0, t_{\bar{n}}]) + (\text{length } g_{q_{\bar{n}}} [t_{\bar{n}}, l_{\bar{n}}]) = d(y_{\bar{n}}, q).$$

This yields $d(y_{\bar{n}}, x_{\bar{n}}) + d(x_{\bar{n}}, p) = d(y_{\bar{n}}, q) = d(y_{\bar{n}}, p)$. Thus $d(y_{\bar{n}}, x_{\bar{n}}) + d(x_{\bar{n}}, p) = (\text{length } g_{q_{\bar{n}}} [0, t_{\bar{n}}]) + (\text{length } g_{p_0} [\bar{t}, l]) = d(y_{\bar{n}}, p)$. Therefore $a_{\bar{n}} := g_{q_{\bar{n}}} [0, t_{\bar{n}}] \cup g_{p_0} [\bar{t}, l]$ is a

minimal join from $y_{\bar{n}}$ to p . We have $g_{q_{\bar{n}}} [0, t_{\bar{n}}] \cap g_{p_{\bar{n}}} [0, l] = \{y_{\bar{n}}\}$. Hence $a_{\bar{n}}$, $g_{p_{\bar{n}}}$ are two (as point sets) distinct minimal joins from $y_{\bar{n}}$ to p , a contradiction. It remains to discuss the possibility (6.2') $\bar{t} > t_0$. Clearly

$$d(q, x_{\bar{n}}) \leq d(p, x_{\bar{n}}) \text{ because}$$

$$d(q, x_{\bar{n}}) + d(x_{\bar{n}}, y_{\bar{n}}) = \text{length}(g_{q\bar{n}}) = d(\{p, q\}, y_{\bar{n}}) \leq d(p, x_{\bar{n}}) + d(x_{\bar{n}}, y_{\bar{n}}).$$

Now the case $d(q, x_{\bar{n}}) = d(p, x_{\bar{n}})$ has already been settled above. Therefore let us assume (6.2) $d(q, x_{\bar{n}}) < d(p, x_{\bar{n}})$.

Condition (6.2') implies $x_{\bar{n}} \in g_{p_0}]t_0, 1]$. Thus

$$(6.3) \quad d(x_0, x_{\bar{n}}) + d(x_{\bar{n}}, p) = d(x_0, p) = 1 - t_0 = d(x_0, q).$$

Hence using (6.2) and (6.3) we get

$$d(q, x_{\bar{n}}) + d(x_{\bar{n}}, x_0) < d(p, x_{\bar{n}}) + d(x_{\bar{n}}, x_0) = d(q, x_0).$$

Thus we have $d(q, x_{\bar{n}}) + d(x_{\bar{n}}, x_0) < d(q, x_0)$, a contradiction.

Clearly using the same arguments as above one can show that the assumption $g_{p\bar{n}} \cap g_{q_0} \neq \emptyset$ also yields a contradiction.

Therefore lemma 4.3 is proved if we can show T. We will do this now.

Proof of T: The proof of T uses considerations in a neighbourhood of the point $x_0 = c_{p_0}(t_0)$. Therefore we start now with a description of this neighbourhood and we introduce notations. Choose $\epsilon > 0$ so small that (6.4) $B_\epsilon(x_0) := \{x \in M / d(x, x_0) \leq \epsilon\}$ is contained in a geodesically convex ball in \hat{M} and (6.5) $B_\epsilon(x_0) \cap \{y_0, p, q\} = \emptyset$.

The point x_0 has a neighbourhood $U(x_0) \subset B_{\frac{\epsilon}{2}}(x_0)$ such that

$$U(x_0) \text{ is homeomorphic to the Euclidean half disc } H_1 := \{(u, v) \in \mathbb{R}^2 / v \geq 0, u^2 + v^2 \leq 1\}$$

and $U(x_0) \cap \partial M$ is homeomorphic to $\{(u, v) \in H_1 / v = 0\}$, (u, v) being Euclidean coordinates in \mathbb{R}^2 . We identify the points in $U(x_0)$ with

$$\text{the corresponding points in } H_1. \text{ Let } d_1 := \{(u, v) \in \mathbb{R}^2 / v = 0, -1 \leq u \leq 0\} \text{ and } d_2 := \{(u, v) \in \mathbb{R}^2 / v = 0, 0 \leq u \leq 1\}.$$

We introduce polar coordinates (r, φ) with $(0, \varphi) = 0 = x_0$

$$\text{and } d_2 = \{(r, \varphi) / \varphi = 0, 0 \leq r \leq 1\}.$$

$$\text{Thus } d_1 = \{(r, \varphi) / \varphi = \pi, 0 \leq r \leq 1\} \text{ and}$$

$H_1 = \{(r, \varphi) / 0 \leq r \leq 1, 0 \leq \varphi \leq \pi\}$. Due to (6.5) we can define

$$\gamma_1 := \min\{\gamma > 0 / |g_{p_0}(t_0 - \gamma)| = 1\}, \quad \gamma_2 := \min\{\gamma > 0 / |g_{p_0}(t_0 + \gamma)| = 1\}$$

$$\gamma_3 := \min\{r > 0 / |g_{q_0}(t_0 + \gamma)| = 1\}, \quad |\cdot| \text{ the Euclidean norm.}$$

Let $y'_0 := g_{p_0}(t_0 - \gamma_1)$, $p' := g_{p_0}(t_0 + \gamma_2)$, $q' := g_{q_0}(t_0 + \gamma_3)$.
We abbreviate $b_1 := g_{p_0}[t_0 - \gamma_1, t_0]$, $b_2 := g_{p_0}[t_0, t_0 + \gamma_2]$.
 $b_3 := g_{q_0}[t_0, t_0 + \gamma_3]$ and $A := \{x \in H_1 \mid r(x) = 1\}$.

We want to show now first the statement T_1 : "The point y'_0 is not contained in the subarc \tilde{A} of A , \tilde{A} joining p' with q' ."

Proof of T_1 : The proof will be indirect. We assume that $y'_0 \in \tilde{A}$. Now $b := b_2 \cup b_3 \cup \tilde{A}$ describes obviously a simple closed curve contained in H_1 . We know by remark 6.2 that b is boundary of a simply connected set $B \subset H_1$, B homeomorphic to the unit disc $D := \{x \in \mathbb{R}^2 \mid |x| \leq 1\}$. We show now first:

$$(6.6) \quad b_1 \setminus \{x_0, y'_0\} \subset B \setminus \partial B$$

Proof of (6.6): We know that $y'_0 \neq p'$, and $y'_0 \neq q'$. Thus y'_0 is an interior point of the arc \tilde{A} hence $d(y'_0, b_2 \cup b_3) > 0$. Therefore there exists a small Euclidean ball $E_\alpha(y'_0)$ with center y'_0 and radius α such that (6.7) $E_\alpha(y'_0) \cap \partial B = E_\alpha(y'_0) \cap \tilde{A}$. Clearly $E_\alpha(y'_0)$ contains points of the two components $B \setminus \partial B$, $\mathbb{R}^2 \setminus B$ of $\mathbb{R}^2 \setminus b$ because b is the frontier of both components. Thus $E_\alpha(y'_0) \cap B \setminus \partial B \neq \emptyset$. Hence let (6.8) $z_0 \in E_\alpha(y'_0) \cap (B \setminus \partial B)$. Then $|z_0| < 1$ because of (6.7) and as $B \subset H_1$. Now every point $y \in (E_\alpha(y'_0) \cap (H_1 \setminus \tilde{A}))$ can be joined with z_0 by an

Euclidean segment $s_y \in E_\alpha(y'_0)$ with $s_y \cap \tilde{A} = \emptyset$.

Therefore by (6.7), (6.8) and (J) all points $y \in E_\alpha(y'_0)$ with $|y| < 1$ belong to $E_\alpha(y'_0) \cap (B \setminus \partial B)$. Thus we get

(6.9) $E_\alpha(y'_0) \cap (B \setminus \partial B) = \{y \in E_\alpha(y'_0) \mid |y| < 1\}$ because $B \setminus \partial B \subset H_1$. Now the continuity of b_1 yields

$b_{1\alpha} := (b_1 \setminus \{y'_0\}) \cap E_\alpha(y'_0) \neq \emptyset$; hence say $z_\alpha \in b_{1\alpha}$.

By the definition of γ_1 we have for all points

$z \in b_1 \setminus \{y'_0\}$ that $|z| < 1$. Therefore using (6.9) we get (6.10) $z_\alpha \in B \setminus \partial B$. We need also (6.11)

$(b_1 \setminus \{x_0, y'_0\}) \cap \partial B = \emptyset$, which holds as $b_1 \cap \partial B = b_1 \cap (\tilde{A} \cup b_2 \cup b_3)$ and $b_1 \cap \tilde{A} = y'_0$, $b_1 \cap (b_2 \cup b_3) = x_0$.

Now all points of $b_1 \setminus \{x_0, y'_0\}$ can be joined with $z_\alpha \in (B \setminus \partial B)$ by a subpath of $b_1 \setminus \{x_0, y'_0\}$. Therefore using (6.10), (6.11) and (J) we get (6.6)

$$b_1 \setminus \{x_0, y'_0\} \subset B \setminus \partial B.$$

Now $B \subset H_1$ therefore $B \setminus \partial B =: \mathring{B} \subset \mathring{H}_1 =: H_1 \setminus \partial H_1$.

Thus (6.6) yields (6.12) $b_1 \setminus \{x_0, y'_0\} \subset H_1 \setminus \partial H_1$.

Hence (6.13) $b_1 \setminus \{x_0, y'_0\} \subset M \setminus \partial M$. Obviously (6.14)

$d(\tilde{A}, x_0) := 10\beta > 0$. Now $b'_1 := g_{p_0}[t_0 - \beta, t_0]$ is a geodesic segment because we have $g_{p_0}[t_0 - \beta, t_0] \setminus \{g_{p_0}(t_0)\} \subset$

$b_1 \setminus \{x_0, y'_0\} \subset M \setminus \partial M$ due to (6.13) and (6.14). Let

$\bar{g}_{p_0}[t_0 - \beta, t_0 + \beta]$ be the (unique) geodesic extension of b'_1

by length β beyond x_0 . At least one of the points

$g_{p_0}(t_0 + \beta)$, $g_{q_0}(t_0 + \beta)$ is different from $\bar{g}_{p_0}(t_0 + \beta)$; say

$g_{p_0}(t_0 + \beta) \neq \bar{g}_{p_0}(t_0 + \beta)$. Denote with \bar{b}_1 the (unique

minimal) geodesic segment from $g_{p_0}(t_0-\beta)$ to $g_{p_0}(t_0+\beta)$. Clearly due to (6.6) and (6.14) $g_{p_0}(t_0-\beta) \in B \setminus \partial B$. Therefore the (normalized) segment \bar{b}_1 must meet ∂B the first time at some point \bar{x} , where \bar{x} may coincide with $g_{p_0}(t_0+\beta)$.

Using (6.14) we have $d(\tilde{A}, \bar{b}_1) > 4\beta$. Hence $\bar{x} \in \partial B \setminus \tilde{A} \subset b_2 \cup b_3$.

Thus let e.g. \bar{x} be a point in b_3 say $\bar{x} = g_{q_0}(t_0+\beta)$.

Denote with \vec{b}_1 the geodesic subsegment of \bar{b}_1 which joins the point $g_{p_0}(t_0-\beta) = g_{q_0}(t_0-\beta)$ with $\bar{x} = g_{q_0}(t_0+\beta)$.

Now the minimal geodesic segment $\vec{b}_1 \subset B \subset M$ is obviously different from $g_{q_0}[t_0-\beta, t_0+\beta]$ because $g_{q_0}(t_0) \notin \vec{b}_1$. This is a contradiction against the minimal (length) property of g_{q_0} . The case that $\bar{x} \in b_2$ can be treated in the same way. Thus we have proved I_1 .

Using I_1 we finish now the proof of I :

We know by I_1 that there are two possibilities:

(6.15) The point p' is contained in the subarc \tilde{A}_1 of A , \tilde{A}_1 joining y_0' with q' .

(6.16) The point q' is contained in the subarc \tilde{A}_2 of A , \tilde{A}_2 joining y_0' with p' .

Let us treat now the case described by (6.15). We want to prove that there exists a natural number \bar{n} such that $g_{p_n} \cap g_{p_0} \neq \emptyset$. For all natural numbers $n \geq \bar{n}$.

Clearly $\varphi(y_0') \neq \varphi(q')$. Let us assume

$$(6.17) \quad \varphi(y_0') > \varphi(q')$$

Then (6.15) implies (6.17) $\varphi(y_0') > \varphi(p') > \varphi(q')$.

Using the same arguments as above in the proof for

(6.6) we find that $b_2 \setminus \{p', x_0\} \subset H_1 \setminus \partial H_1$.

Let $\bar{A}_1 := \{x \in E_1 / r(x) = 1, \varphi(p') \leq \varphi(x) \leq \frac{5}{4}\pi\}$, with

$E_1 := E_1(x_0) := \{x \in \mathbb{R}^2 / |x| \leq 1\}$ and $d_1^1 := \{x \in E_1 / \varphi(x) = \frac{5}{4}\pi, 0 \leq r(x) \leq 1\}$.

Now $\bar{b} := d_1^1 \cup b_2 \cup \bar{A}_1$ is a closed curve which is contained in $E^1 := E_1 \setminus \{y \in E_1 / r(y) = 1, 0 \leq \varphi(y) < \varphi(d^1)\}$.

The curve \bar{b} is simple because $(b_2 \setminus \{d^1, x_0\}) \cap (\partial E_1 \cup d_1^1) \neq \emptyset$. Hence \bar{b} is boundary of a simply connected subset $B_2 \subset E^1$.

Now since $B_2 \subset E^1$ the point q^1 is not contained in B_2 . Thus we have $d(B_2, q^1) > 0$. Therefore there exists a number

$\bar{\gamma}_3 \in]0, \gamma_3[$ such that (6.18) $d(g_{q_0}(t_0 + \bar{\gamma}_3), B_2) > 0$.

Clearly $x^1 := g_{q_0}(t_0 + \bar{\gamma}_3) \in E^1 \setminus A$ by the definition of γ_3 .

Therefore and because of (6.18) there exists a number $\bar{\epsilon} > 0$ such that

$$(6.19) \quad B_{\bar{\epsilon}}(x^1) := \{z \in M / d(x^1, z) \leq \bar{\epsilon}\} \subset E_1 \setminus (B_2 \cup A).$$

Now since \bar{b} is simple there exists a number $\eta > 0$ such that $E_\eta(y_0^1) \cap \partial B_2 \subset \bar{A}_1$. By arguments similar to those above for the proof (6.9) we get

$$B_\eta := E_\eta(y_0^1) \cap B_2 \setminus \partial B_2 = \{x \in E_\eta(y_0^1) / |x| < 1\}.$$

Therefore it is obvious that for every point $z \in B_\eta$ there exists a number $\delta > 0$ ¹⁾, such that $B_\delta(z) :=$

$$\{z' \in M / d(z', z) \leq \delta\} \subset B_2.$$

Now because of the continuity of g_{q_0} there exists a number $\bar{\gamma}_4 \in]0, \gamma_4[$ with $\tilde{x} := g_{q_0}(t_0 + \bar{\gamma}_4) \in E_\eta(y_0^1)$. Thus the definition of γ_4 yields $\tilde{x} \in \{x \in E_\eta(y_0^1) / |x| < 1\}$. Hence $\tilde{x} \in B_\eta$ and we have a number $\bar{\delta} > 0$ such that (6.20)

$$B_{\bar{\delta}}(\tilde{x}) \subset B_2.$$

1) δ is depending on the point z .

The definition of \bar{y}_1, \bar{y}_3 yields $g_{q_0}(I) \subset E_1 \setminus (A \cup (d'_1 \setminus \{x_0\}))$

if $I := [t_0 - \bar{y}_1, t_0 + \bar{y}_3]$. Therefore it is easy to see that

there exists a positive number $\omega < \min\{\varepsilon, \bar{\delta}\}$ such that

$$(6.20) \quad G_\omega := \bigcup_{t \in I} B_\omega(g_{q_0}(t)) \subset E_1 \setminus (A \cup (d'_1 \setminus \{x_0\})).$$

We know from the beginning of the proof of lemma 6.3

that the sequence $g_{q_n}(I)$ converges uniformly against

$g_{q_0}(I)$. Therefore there exists a natural number \bar{n} such that

we have for all $n \geq \bar{n}$

$$(6.21) \quad g_{q_n}(t_0 - \bar{y}_1) \in B_{\bar{\delta}}(\bar{x}),$$

$$(6.22) \quad g_{q_n}(t_0 + \bar{y}_3) \in B_{\bar{\varepsilon}}(x'),$$

$$(6.23) \quad g_{q_n}(I) \subset G_\omega.$$

The combination of (6.20) and (6.21) yields $g_{q_n}(t_0 - \bar{y}_1) \in B_2$.

Further combining (6.19) and (6.22) yields $g_{q_n}(t_0 + \bar{y}_3) \notin B_2$.

Therefore using (J) we get $g_{q_n}(I) \cap \partial B_2 \neq \emptyset$. This implies

together with (6.20) and (6.23) that $g_{q_n}(I) \cap b_2 \neq \emptyset$.

Thus $g_{q_n} \cap g_{p_0} \neq \emptyset$ for all $n > \bar{n}$.

The preceding considerations proved T in the subcase

(6.17) of (6.15). However it is obvious that those

arguments can be used also to prove T in the subcase

$\varphi(y_0') < \varphi(d)$ of (6.15).

Further it is clear that the procedure used to prove T

in case (6.15) can be applied to prove T in case (6.16).

In case (6.16) the above procedure yields the existence

of a number $\tilde{n} \in \mathbb{N}$ such that for all $n \in \mathbb{N}$ with $n \geq \tilde{n}$ is

$g_{p_n} \cap g_{q_0} \neq \emptyset$. Thus T is shown. This completes the

proof of lemma 6.3.

The subsequent proposition contains already a certain amount of information about the structure of the cut locus of two distinct points in a simply connected space of type (**).

Proposition 6.3: Let p, q be any two distinct points in a simply connected space (M, d) of type (**), M being sub-surface of a space (\hat{M}, \hat{d}) of type (*). Then:

a) The set $C_{\{p, q\}} \setminus \partial M \neq \emptyset$, $C_{\{p, q\}}$ the cut locus of the set $\{p, q\}$.

Let K be any connected component of $C_{\{p, q\}} \setminus \partial M$ then the following statements are valid:

b) For every $x \in K$ exists a number $\varepsilon > 0$ such that

$P_x := \overset{\circ}{B}_\varepsilon(x) \cap K = \overset{\circ}{B}_\varepsilon(x) \cap A(p, q)$ and P_x is C^1 -diffeomorphic to $]0, 1[$, $\overset{\circ}{B}_\varepsilon(x) := \{x' \in M / d(x, x') < \varepsilon\}$.

c) Let $\alpha(t) :]0, 1[\rightarrow M$ be any C^1 -smooth embedding

with $\alpha(]0, 1[) = P_x$; (the existence of such an embedding is assured by b). Then every point $\alpha(t) \in P$ is initial

point of exactly two different normalized minimal

joins $g_{pt}(s)$, $g_{qt}(s)$ going to the points p, q

respectively. The tangent vector $\dot{\alpha}(t) := \frac{d\alpha(t)}{dt}$ bisects the

(nonzero) angle built by the distinct initial vectors:

$\dot{g}_{pt}, \dot{g}_{qt}$ of the paths $g_{pt}(s), g_{qt}(s)$ respectively.

d) The set K is a submanifold of M and K is C^1 -diffeomorphic to the intervall $]0, 1[$.

e) Let $\Psi(t) :]0, 1[\rightarrow M$ be any C^1 -smooth embedding with

$\Psi(]0, 1[) = K$; (the existence of such an embedding is

guaranteed by d). Now let (t_n) be any sequence in $]0,1[$ with $\lim t_n = 0$ or $\lim t_n = 1$. Then the sequence $\Psi(t_n)$ has no cluster point in $M \setminus \partial M$.

Proof of proposition 6.3: Statement a) is an immediate consequence of proposition 6.2.

Proof of b): We know by a) that $K \neq \emptyset$. Let x be any point in K . Then by the definition of the cut locus, the point x is limit of a sequence of picas relative to the set $\{p,q\}$. Therefore by Lemma 6.3 the point x must be also a pica relative to $\{p,q\}$. Thus we have by Lemma 6.3 and theorem 6.1 exactly two normalized minimal joins $g_{px}(s), g_{qx}(s)$ from x to the set $\{p,q\}$. The paths $g_{px}(s), g_{qx}(s)$ are going from x to p,q respectively; this follows again by theorem 6.1. Clearly $x \in A(p,q)$ and $A(p,q) := \{y \in M / f(y) = 0\}$ with $f : M \rightarrow \mathbb{R}$ defined by $f(y) := d(p,y) - d(q,y)$ for all $y \in M$. We know by theorem 6.2 that f is C^1 -smooth in a neighbourhood $U(x) \subset M \setminus \partial M$ of the point x because $x \in M \setminus (\partial M \cup \{p,q\})$. Obviously at the point x

$$\text{grad } f(x) = \text{grad } d(p,x) - \text{grad } d(q,x) = -\dot{g}_{px} + \dot{g}_{qx} \neq 0$$

because x is a pica; $\dot{g}_{px}, \dot{g}_{qx}$ the initial vectors of the normalized paths $g_{px}(s), g_{qx}(s)$ respectively. Using the implicit function theorem cf. [33] p. 98 it is easily seen that there exists a number $\varepsilon > 0$ such that

$$P_x := A(p,q) \cap \overset{\circ}{B}_\varepsilon(x) \text{ is } C^1\text{-diffeomorphic to }]0,1[,$$

$$\overset{\circ}{B}_\varepsilon(x) := \{x' \in M / d(x,x') < \varepsilon\} \subset M \setminus \partial M. \text{ Thus say we have}$$

a C^1 -smooth embedding $\alpha(t) :]0,1[\rightarrow M \setminus \partial M \subset M$ with

$\alpha(]0,1[) = P_x$ and $(\text{grad } f)(\alpha(t)) \neq 0$ for all $t \in]0,1[$.

Therefore all points in $\alpha(]0,1[)$ are picas and we have obviously

$$P_x = \overset{\circ}{B}_\varepsilon(x) \cap \alpha(]0,1[) = \overset{\circ}{B}_\varepsilon(x) \cap K = \overset{\circ}{B}_\varepsilon(x) \cap C_{\{p,q\}};$$

recall $A(p,q) \supset C_{\{p,q\}}$. This proves b).

Proof of c): We know already $P_x = \overset{\circ}{B}_\varepsilon(x) \cap \text{Picas}\{p,q\}$

with $\text{Picas}\{p,q\} := \{y \in M / y \text{ is pica relative to the set}$

$\{p,q\}$. Again by theorem 6.1 every point $\alpha(t) \in P_x$ is initial

point of exactly two normalized minimal joins $g_{pt}(s), g_{qt}(s)$

to the set $\{p,q\}$, the paths $g_{pt}(s), g_{qt}(s)$ going from

$\alpha(t)$ to p, q respectively. We have $f(\alpha(t)) = 0$ for all

$t \in]0,1[$. Therefore

$$\langle \text{grad } f(\alpha(t)), \dot{\alpha}(t) \rangle = \langle \text{grad } d(p, \alpha(t)) - \text{grad } d(q, \alpha(t)), \dot{\alpha}(t) \rangle =$$

$$\langle \dot{g}_{qt}(0) - \dot{g}_{pt}(0), \dot{\alpha}(t) \rangle = 0, \quad \langle , \rangle \text{ the scalar product}$$

induced by the Riemannian metric on the tangent space

$$T_{\alpha(t)} \hat{M}.$$

Thus we have

$$(6.24) \quad \langle \dot{g}_{pt}(0), \dot{\alpha}(t) \rangle = \langle \dot{g}_{qt}(0), \dot{\alpha}(t) \rangle .$$

We know

$$(6.25) \quad \langle \dot{g}_{pt}(0), \dot{g}_{pt}(0) \rangle = 1 = \langle \dot{g}_{qt}(0), \dot{g}_{qt}(0) \rangle$$

and

$$(6.26) \quad \dot{\alpha}(t) \neq 0, \quad \dot{g}_{pt}(0) \neq \dot{g}_{qt}(0) .$$

Now using (6.24), (6.25), (6.26) and the fact that the

tangent space $T_{\alpha(t)} \hat{M}$ is two-dimensional it is easily seen

that that $\dot{\alpha}(t)$ bisects the (nonzero) angle built by the

(distinct) vectors $\dot{g}_{pt}(0), \dot{g}_{qt}(0)$ at the point $\alpha(t)$.

This proves c).

Proof of d): We have already proved in c) that for every point $x \in K$ there exists a neighbourhood $B_\varepsilon(x)$ such that $B_\varepsilon(x) \cap K$ is diffeomorphic to $]0,1[$. Therefore and by the definition of K , K is a connected one-dimensional C^1 -smooth submanifold of $M \setminus \partial M \subset M$. According to the classification theorem of one-dimensional manifolds (see e.g. [52] p. 55) K is either diffeomorphic to the unit circle S^1 or to the open unit interval $]0,1[$, because every point $x \in K$ has a neighbourhood being diffeomorphic to $]0,1[$, cf. [52] p. 55. We want to prove that K cannot be diffeomorphic to S^1 . Assume the contrary. Then K being C^1 -diffeomorphic to S^1 is a simple closed curve contained in M . Now M is subsurface of an unbordered complete Riemannian manifold \hat{M} , \hat{M} diffeomorphic to \mathbb{R}^2 . Let B_k be the bounded component of $\hat{M} \setminus K$. By remark 6.2 we have

$$(6.27) \quad B_k \text{ is contained in } (M \setminus (\partial M \cup K)).$$

Clearly by (J) we have $K = \partial \bar{B}_k$, \bar{B}_k the closure of B_k and B_k is homeomorphic to the open unit disc,

$$D := \{(u,v) \in \mathbb{R}^2 / u^2 + v^2 < 1\}.$$

Further we have

$$(6.28) \quad M \setminus (B_k \cup K) \neq \emptyset.$$

Namely in case $\partial M = \emptyset$ we get $M = \hat{M}$. Thus here $M \setminus (B_k \cup K)$ is homeomorphic to $\mathbb{R}^2 \setminus D$ ¹⁾. Hence (6.28) is obviously true.

1) $D := \{(u,v) \in \mathbb{R}^2 / u^2 + v^2 \leq 1\}$

In case $\partial M \neq \emptyset$ we have $\partial M \cap \bar{B}_k = \emptyset$ for $K \cap \partial M = \emptyset$ and (6.27). Thus (6.28) holds here too. In order to derive a contradiction we discuss now all possibilities for the location of the points p, q . Clearly $\{p, q\} \cap K = \emptyset$ because $p \neq q$. First assume say $\{p, q\} \subset B_k$. Now by (6.28) exists a point $y \in M \setminus \bar{B}_k$. Let

$g_y = g_y([0, d(\{p, q\}, y)])$ be any normalized minimal join from y to $\{p, q\}$. By (J) we get

$$g_y([0, d(\{p, q\}, y)]) \cap K = g_y([0, d(\{p, q\}, y)]) \cap (\text{Picas}\{p, q\}) \cap (M \setminus \partial M) \neq \emptyset.$$

This yields easily a contradiction against the minimal length property of g_y . Using the same arguments we can exclude that $\{p, q\} \subset M \setminus \bar{B}_k$. It remains to discuss the possibilities

$$(6.29) \quad q \in M \setminus \bar{B}_k \quad \text{and} \quad p \in B_k$$

$$(6.29') \quad p \in M \setminus \bar{B}_k \quad \text{and} \quad q \in B_k.$$

It is sufficient to treat case (6.29) because case (6.29') can be reduced to (6.29) by simply changing the notations. We consider now case (6.29). By theorem 6.2 the map $\text{grad } d(q, \cdot)$ defines a continuous vector field without any zero on $M \setminus (\partial M \cup \{q\})$. Thus by (6.27) we have that $\text{grad } d(q, \cdot)$ defines a continuous vector field without any zero on $\bar{B}_k \subset M \setminus (\partial M \cup \{q\})$. Therefore it is wellknown that the rotation index of the vector field $\text{grad } d(q, \cdot)$ on the boundary curve $\partial \bar{B}_k = K$ must be zero, see e.g. [46] p. 16, ¹⁾

1) Recall that the following is valid: "Let B be any compact bordered topological subsurface of \mathbb{R}^2 . Assume that

On the other hand it is well known, too:

(6.29) "If the rotation index of a vector field V (here is $V = \text{grad } d(q, \cdot)$) on a closed, simple and smooth curve K is different from 1 then exists at least one point $y \in K$ where the field vector is parallel with the tangent at K in y "; see e.g. [46] p. 25, Satz 4.5.

However in our situation (6.29) is impossible because by proposition 6.3 c) the tangent at K must bisect the (nonzero) angle between the vectors $\text{grad } d(p, \cdot)$, $\text{grad } d(q, \cdot)$ at all points $y \in K$. This is a contradiction. Therefore K cannot be diffeomorphic to S^1 . Thus K must be diffeomorphic to the open unit interval $]0, 1[$. This proves d).

Proof of e): The statement e) is essentially a consequence of Lemma 6.3 and has been shown more or less already in the proof of b). Let (t_n) be a sequence in $]0, 1[$ with say $\lim t_n = 1$ and say $\lim \Psi(t_n) = x_0 \in M \setminus \partial M$. Then by definition x_0 is not contained in the connected

Cont. footnote 1) from p. 175:

we have a vector field V on B and there exist only finitely many singular points $\{p_1, \dots, p_n\} =: S$ i.e. where V is not continuous or where V has a zero, or where V is not defined. Now if $S \cap \partial B = \emptyset$, then the algebraic number of singular points of V on B equals the rotation index of the field V on ∂B ." See e.g. [46] p. 18.

component $K = \Psi]0,1[$ of $C_{\{p,q\}} \setminus \partial M$. However it has been shown in the proof of b) that $x_0 \in M \setminus \partial M$ being limit of picas in K must belong to $B_\varepsilon(x_0) \cap K$, a contradiction. This proves e). Hence proposition 6.3 is completely shown.

Using Lemma 6.3 and the preceding proposition we give in the following theorem a detailed description of the cut locus of two distinct points in a simply connected space of type (**).

Theorem 6.3: Let (M,d) be a simply connected space of type (**) and let p,q be any two distinct points in M .

We define the point $m \in M$ by the condition

$$d(p,m) = d(m,q) = \frac{1}{2} d(p,q); \text{ and we abbreviate}$$

$$I_1 :=]0,1[\text{ and } I_{-1} :=]-1,0[.$$

If $m \notin \partial M$, then we have the following results:

a₁) Let K be the connected component of ¹⁾ $C_{\{p,q\}} \setminus \partial M$ with $m \in K$. Then there exists a C^1 -smooth embedding

$$\Psi: (I_{-1} \cup I_1) \rightarrow M, \text{ with } \Psi(0) = m \text{ and } \Psi(I_{-1} \cup I_1) = K.$$

Further for every $\Psi(I_k), k \in \{-1,1\}$ exactly one of the two subsequent alternative possibilities (F) or (Q) must hold:

1) $C_{\{p,q\}}$ is the cut locus relative to the set $\{p,q\}$.

Later on in a₂) we state that there exists only one component of $C_{\{p,q\}} \setminus \partial M$. We use this formulation of a₁) for technical reasons because we prove first this weaker statement a₁).

(Γ) For every sequence t_n in I_k with $\lim t_n = k$ the sequence $d(m, \Psi(t_n))$ is unbounded.

(Ω) There exists a point $x_k \in \partial M \setminus \{m\}$ such that for every sequence (t_n) in I_k with $\lim t_n = k$ the sequence $\Psi(t_n)$ converges to x_k .

a₂) We have $\Psi(I_{-1} \cup I_1) = C_{\{p,q\}} \setminus \partial M$, $C_{\{p,q\}} = \overline{\Psi(I_1) \cup \Psi(I_{-1})}$ and $\overline{\Psi(I_{-1})} \cap \overline{\Psi(I_1)} = \{m\}$ with $\overline{\Psi(I_1)}$, $\overline{\Psi(I_{-1})}$ the closure of $\Psi(I_1)$, $\Psi(I_{-1})$ respectively.

If $m \in \partial M$ then we have the following results:

b₁) There exists a continuous embedding $\tilde{\Psi}: I_1 \rightarrow M$ with $\tilde{\Psi}(0) = m$ and the restriction $\tilde{\Psi}:]0,1[\rightarrow M$ is a C^1 -smooth embedding with $(\tilde{\Psi}]0,1[) = C_{\{p,q\}} \setminus \partial M$.

b₂) For $\tilde{\Psi}(I_1)$ exactly one of the above statements (Γ) or (Ω) must hold. We have $C_{\{p,q\}} = \overline{\tilde{\Psi}(I_1)}$.

Let $m \in \partial M$ or $m \in M \setminus \partial M$. Then the following results hold:

c) The cut locus $C_{\{p,q\}}$ agrees with the set of all "generalized picas" relative to $\{p,q\}$, i.e. every generalized pica is a limit of picas. ¹⁾

1) This result is a converse of Lemma 6.3!

d) There exists a number $\lambda > 0$ such that

$$B_\lambda(m) \cap A(p,q) = B_\lambda(m) \cap C_{\{p,q\}},$$

with $B_\lambda(m) := \{y \in M / d(m,y) \leq \lambda\}$.

Proof of theorem 6.3: We prove first theorem 6.3 a₁):

For this let $m \notin \partial M$. Clearly $m \in C_{\{p,q\}} \setminus \partial M$. Let K be the connected component of $C_{\{p,q\}} \setminus \partial M$ with $m \in K$. We know by proposition 6.3 b,d) that we have a C^1 -smooth embedding $\Psi: I_{-1} \cup I_1 \rightarrow M$ with $\Psi(0) = m$ and $\Psi(I_{-1} \cup I_1) = K$. Let $k \in \{-1,1\}$. We assume that there exists a sequence (t_n) in I_k with $\lim t_n = k$ such that $d(\Psi(0), \Psi(t_n))$ is bounded. Then $(\Psi(t_n))$ has a cluster point $x_k \in M$. By proposition 6.3 e) we have $x_k \in \partial M$. Now by proposition 6.3 c) all points $\Psi(t_n)$ are picas relative to $\{p,q\}$. Therefore Lemma 6.3 and theorem 6.1 imply that there exists a minimal join g_1 from x_k to p and a minimal join g_2 from x_k to q with

$$(6.30) \quad g_1 \cap g_2 = \{x_k\}.$$

We assume that $g_1(t), g_2(t) : [0, \bar{r}] \rightarrow M$, $\bar{r} = d(x_k, p) = d(x_k, q)$ are normalized paths. Let $\bar{g}(t) : [0, r_1] \rightarrow M$ be the normalized minimal join from p to q , then $r_1 = d(p, q)$. We define

$$\omega_1 := \max \{t \in [0, r_1] / \bar{g}(t) \in g_1\},$$

$$\omega_2 := \min \{t \in [0, r_1] / \bar{g}(t) \in g_2\}.$$

Now we prove

$$(6.31) \quad \omega_1 < \frac{r_1}{2} < \omega_2.$$

For this note: If $\omega_1 > 0$ then

$g_1[0, \bar{r}] = g_1[0, \bar{r} - \omega_1] \cup \bar{g}[0, \omega_1]$ for in case
 $g_1[\bar{r} - \omega_1, \bar{r}] \neq \bar{g}[0, \omega_1]$ the uniqueness of minimal joins
 is violated. This yields $\omega_1 \leq \frac{r_1}{2}$. Namely otherwise
 the path $g_1[0, \bar{r} - \omega_1] \cup \bar{g}[\omega_1, r_1]$ going from x_k to q
 has a length shorter than the join $g_1[0, \bar{r} - \omega_1] \cup \bar{g}[0, \omega_1]$
 from x_k to p , a contradiction because $d(x_k, p) = d(x_k, q)$.
 We have to exclude that $\omega_1 = \frac{r_1}{2}$. Now if $\omega_1 = \frac{r_1}{2}$ then
 $d := g_1[0, \bar{r} - \omega_1] \cup \bar{g}[\omega_1, r_1]$ is a minimal join from
 x_k to q because $\bar{d} := g_1[0, \bar{r} - \omega_1] \cup \bar{g}[0, \omega_1]$ is a minimal
 join from x_k to p and length $d = \text{length } \bar{d}$. Therefore
 theorem 6.1 implies $g_1[0, \bar{r} - \omega_1] = g_2[0, \bar{r} - \omega_1] \supset \{x_k, \bar{g}(\omega_1)\}$
 $= \{x_k, m\} \neq \{x_k\}$, because $x_k \in \partial M$, $m \notin \partial M$. This yields a
 contradiction against (6.30). Using the same arguments we
 can show that $\frac{r_1}{2} < \omega_2$. This proves (6.31).

Let $b := g_1[0, \bar{r} - \omega_1] \cup \bar{g}[\omega_1, \omega_2] \cup g_2[0, \bar{r} + \omega_2 - r_1]$.

Clearly b is a simple closed curve in M . By remark 6.2
 the curve b is boundary of a simply connected set $A \subset M$,
 A being homeomorphic to the closed unit disc. By proposition
 6.3 c) it is obvious that Ψ is transversal to b at
 $\Psi(0) = m = \bar{g}(\frac{r_1}{2})$. Therefore it is easy to see that there
 exists a small number $\delta > 0$ such that either $a_1 := \Psi]0, \delta[$
 or $a_1 := \Psi] -\delta, 0[$ are contained in $A \setminus b$.

Let $k' \in \{-1, 1\}$ with $a_{k'} \subset A \setminus b$.

We prove now that $a_{k'} \subset A \setminus b$ implies

$$(6.32) \quad c_{k'} := \Psi(I_{k'}) \setminus \{\Psi(0)\} \subset A \setminus b.$$

Assume the contrary. Then $c_{k'} \cap b \neq \emptyset$. Let $y_{k'} \in (c_{k'} \cap b)$.

By proposition 6.3 c) the point y_k , is a pica; there exist two distinct (normalized) minimal joins $\tilde{g}_p(t)$, $\tilde{g}_q(t)$ going from y_k , to p, q respectively and \tilde{g}_p, \tilde{g}_q have the same length.

Now $y_k \in b$. Say $y_k \in g_1 [0, \bar{r} - \omega_1]$. Thus let $y_k = g_1(t_k)$. Then $e_1 := g_1[0, t_k] \cup \tilde{g}_q$ is a minimal join from x_k to q . This holds because $e_2 := g_1[0, t_k] \cup \tilde{g}_p$ having the same length as e_1 is a minimal join from x_k to p and because $d(x_k, p) = d(x_k, q)$. Therefore (by theorem 6.1) $e_1 = g_2$. Hence $t_k = 0$, as $g_1 \cap g_2 = \{x_k\}$. Thus $y_k \notin g_1[0, \bar{r} - \omega_1]$. Using the same arguments we find that $y_k \notin g_2[0, \bar{r} + \omega_1 - r_1]$. Obviously $y_k \notin \bar{g}[\omega_1, \omega_2] \setminus \{m\}$. Clearly $\Psi(0) = m \neq y_k = \Psi(s)$ because $s \notin]-\delta, \delta[$ and as Ψ is an embedding. Thus $y_k = x_k \in \partial M$, a contradiction because $y_k \in \Psi(I_k) \subset C_{\{p, q\}} \setminus \partial M$. This proves (6.32).

We know now that $c_k \subset A \setminus b$. Let (t_n) be any sequence in I_k , with $\lim t_n = k'$. Then $\Psi(t_n)$ has a cluster point \bar{x} in A and by proposition 6.3 e) $\bar{x} \in \partial M$. Hence $\bar{x} \in A \cap \partial M$. Thus $\bar{x} \in b$ as $A \setminus b = A \setminus \partial A \subset M \setminus \partial M$. We know by Lemma 6.3 that \bar{x} is a generalized pica relative to $\{p, q\}$.

During the proof of (6.32) we have shown:

(6.32') If a generalized pica \tilde{x} is contained in b then

$$\tilde{x} \in \{x_k, m\}$$

This yields here together with $\{x_k, m\} \cap \partial M = \{x_k\}$ that

$$(6.33) \quad \bar{x} = x_k.$$

The next step in our proof will be to show

$$(6.34) \quad c_k = c_k.$$

The proof of (6.34) is long and needs several steps. Therefore we shall explain now first how theorem 6.3 a₁) follows from the preceding considerations using (6.34) Recall it remains to prove that in case $m \notin \partial M$ exactly one of the statements (Γ) or (Ω) holds. Clearly exactly one of the statements (Γ) or $(\neg \Gamma)$ ¹⁾ must hold. It is obvious that (Ω) is not valid in case (Γ) holds. Hence theorem 6.3 is proven if we can show that $(\neg \Gamma)$ implies (Ω) . For this $(\neg \Gamma)$ implies, that there exists a sequence t_n in I_k with $\lim t_n = k$ such that the sequence $d(\Psi(0), \Psi(t_n))$ is bounded. Therefore the sequence $\Psi(t_n)$ has a cluster point $x_k \in M$. By proposition 6.3 e) the point $x_k \in \partial M$. We proved in the preceding considerations up to statement (6.33) that there exists a number $k' \in \{-1, 1\}$ such that for every sequence (t_n) in $I_{k'}$ with $\lim t_n = k'$ we get $\lim \Psi(t_n) = x_k$. Now using (6.34) we get $I_{k'} = I_k$. Thus for every sequence (t_n) in I_k $\lim t_n = k'$ implies $\lim \Psi(t_n) = x_k$. This is the statement of (Ω) . Hence theorem 6.3a₁) is shown using (6.34).

Proof of (6.34): We shall prove: (6.35) "There exists a number $\bar{\delta} > 0$ such that in $B_{\bar{\delta}}(x_k) := \{y \in M / d(y, x_k) \leq \bar{\delta}\}$ all generalized picas relative to $\{p, q\}$ belong to c_k ".

Clearly (6.35) yields (6.34) because we know from the beginning that there exists a sequence $\Psi(t_n)$ in c_k with $\lim \Psi(t_n) = x_k$. The proof of (6.35) uses considerations in

1) The symbol " \neg " denotes the negation of a statement.

a neighbourhood of the point x_k . Therefore we start now with a description of this neighbourhood and we introduce notations:

The point $x_k \in \partial M$ has in the space (M, d) a neighbourhood $U(x_k)$ with $\bar{g}[0, r_1] \cap U(x_k) = \emptyset$ and such that $U(x_k)$ is homeomorphic to $\overset{\circ}{H}_2 := \{(u, v) \in \mathbb{R}^2 / v \geq 0, u^2 + v^2 < 2\}$; under this homeomorphism $U(x_k) \cap \partial M$ is mapped on $\{(u, v) \in \overset{\circ}{H}_2 / v = 0\}$; (u, v) Euclidean coordinates. We identify the points in $U(x_k)$ with the corresponding points in $\overset{\circ}{H}_2$; clearly $x_k = (0, 0)$. Pick $H_1 := \{y \in H_2 / |y| \leq 1\}$, $| \cdot |$ the Euclidean norm.

We can obviously define:

$$\gamma_1 := \min \{ t > 0 / |g_1(t)| = 1 \},$$

$$\gamma_2 := \min \{ t > 0 / |g_2(t)| = 1 \},$$

$$\gamma := \min \{ t > 0 / |\Psi(k' - k't)| = 1 \}.$$

Clearly $z_1 := g_1(\gamma_1)$, $z_2 := g_2(\gamma_2)$, $z := \Psi(k' - k'\gamma)$ are three distinct points. ¹⁾ We introduce also polar coordinates (r, ϕ) for the points in $\underline{E}_2 := \{y \in \mathbb{R}^2 / |y| \leq 2\} \supset H_1$; we define for $y \in E_2$, $r(y) := |y|$ and $\phi(1, 0) := 0$, $\phi(0, 1) := \frac{\pi}{2}$, $\phi(-1, 0) = \pi$.

We prove now first:

(6.36) The point z is contained in the subarc \tilde{a} of
 $a := \{y \in H_1 / |y| = 1\}$, \tilde{a} joining z_1 with z_2 .

1) See also the proof of (6.38)!

Proof of 6.36: Assume the contrary say we have:

$$(6.37) \quad \phi(z_1) > \phi(z_2) > \phi(z)$$

or (6.37') $\phi(z) > \phi(z_1) > \phi(z_2)$, $\phi(z) > \phi(z_2) > \phi(z_1)$,
 $\phi(z_2) > \phi(z_1) > \phi(z)$.

We consider now first case (6.37). We define

$$S := (\{\Psi(t) \in c_k, / |t| \geq |k' - k'\gamma|\} \cup \{x_k\}) ,$$

$$d_1 := \{y \in \mathbb{R}^2 / |y| = 1, 0 \leq \phi(y) \leq \phi(z) \text{ or } \frac{7}{4}\pi \leq \phi(y) \leq 2\pi\}$$

$$d_2 := \{y \in \mathbb{R}^2 / \phi(y) = \frac{7}{4}\pi, 0 \leq r(y) \leq 1\}.$$

Now $h := S \cup d_1 \cup d_2$ is obviously a simple closed curve in $E_1 := \{y \in \mathbb{R}^2 / |y| \leq 1\}$. Therefore h is boundary of a topological disc F , $F \subset E_1$. Pick a sequence of points $(v_n) \in S$, with $|v_n| < \frac{1}{2}$, $v_n \neq x_k$ and $\lim v_n = x_k$. Now every point v_n in this sequence, being a pica, is initial point of two distinct normalized minimal joins

$g_{1n}(t)$, $g_{2n}(t)$ one of them going from v_n to p , the other one going from v_n to q .

It is easily seen that for one of those minimal joins say for g_{1n} exists a number $\tilde{\epsilon} > 0$ such that $g_{1n}[0, \tilde{\epsilon}] \subset F \cdot h$. Recall $S \setminus \{x_k, z\}$ is a C^1 -smooth, one-dimensional manifold and the tangent of S at the point v_n bisects the (nonzero) angle between g_{1n} and g_{2n} . Since $\{p, q\} \cap F = \emptyset$, the path g_{1n} must leave F . Therefore we can define $\gamma_{1n} := \min \{t \geq \tilde{\epsilon} / g_{1n}(t) \in \partial F\}$.

Clearly $g_{1n}(\gamma_{1n}) := z_{1n} \notin d_2$.

We prove now

$$(6.38) \quad z_{1n} \notin S.$$

Assume the contrary, then we have two distinct minimal joins \hat{g}_p, \hat{g}_q going from z_{1n} to p, q respectively.

By using techniques which had been applied in the proof of (6.31) we find that we have two minimal joins

$g_{1n}[0, \gamma_{1n}] \cup \hat{g}_p, g_{1n}[0, \gamma_{1n}] \cup \hat{g}_q$ going from z_{1n} to p, q respectively. One of the paths \hat{g}_p, \hat{g}_q say \hat{g}_q must have the same end point as g_{2n} . Therefore $g_{2n}, g_{1n}[0, \gamma_{1n}] \cup \hat{g}_q$ are two distinct minimal joins with the same end points, a contradiction. This proves (6.38). Hence

$$(6.38') \quad z_{1n} \in d_1 \setminus \{z\}.$$

Now g_{1n} contains a subsequence converging against a normalized minimal join $g_0(t)$ from x_k to $\{p, q\}$, say g_0 ends up in p . Let $a' := \{y \in H_1 / |y| = \frac{1}{2}\}$, and $d(a, a') =: \eta$. Then clearly $\eta > 0$ and $\gamma_{1n} > \eta$ for all $n \in \mathbb{N}$. Obviously $g_0[0, \eta] \subset F$ because $g_{1n}[0, \gamma_{1n}] \subset F$ and as F is closed. Defining $\gamma_0 := \min \{t > 0 / |g_0(t)| = 1\}$ we get ¹⁾

$$(6.39) \quad g_0]0, \gamma_0[\subset F \setminus \partial F \quad \text{and} \quad g_0(\gamma_0) \in d_1 \setminus \{z\}.$$

One can prove (6.39) as follows: Using the considerations from the proof of (6.38') we find that $g_0]0, \eta[\cap (S \cup d_2) = \emptyset$. Thus $g_0]0, \eta[\cap \partial F = \emptyset$ because $\eta < d(x_0, a)$. Hence $g_0]0, \eta[\subset F \setminus \partial F$ as $g_0]0, \eta[\subset F$. Therefore we can de-

1) In the following considerations we show more than we need for the proof of (6.39) and (6.36). However it will be comfortable to refer to those arguments in subsequent parts of the proof.

fine $\bar{\gamma}_0 := \min \{t \geq \eta / g(t) \in \partial F\}$. Clearly $g_0(\bar{\gamma}_0) \in F \setminus \partial F$. Using again the arguments from the proof of (6.38) we find that $g_0(\bar{\gamma}_0) \in d_1 \setminus \{z\}$. Therefore statement (6.39) is shown if we can prove

$$(6.39') \quad \bar{\gamma}_0 = \gamma_0 .$$

Now (6.39') holds if we have

$$(6.40) \quad d_1 = \{y \in F / |y| = 1\}.$$

We want to prove now (6.40). For this let z^* be a point in H_1 with $|z^*| = 1$ and $z^* \notin d_1$. Thus

$$(6.41) \quad z^* \in a \setminus d_1 .$$

For the proof of (6.40) we have to show that (6.41) implies

$$(6.41') \quad z^* \in E_2 \setminus F .$$

It is easily seen that there exists a positive number μ such that

$$(6.42) \quad E_\mu(z^*) \cap \partial F = \emptyset ,$$

with $\underline{E}_\mu(z^*) := \{y \in E_2 / |z^* - y| \leq \mu\}$.

By remark 6.2 is $F \subset E_1$. Thus defining $G := \{y \in E_\mu(z^*) / |y| > 1\}$ we get $G \subset E_2 \setminus F$; clearly $G \neq \emptyset$ and $G \subset E_\mu(z^*)$.

This yields using (6.42) and (J) that

$$(6.41) \quad \underline{E}_\mu(z^*) \subset E_2 \setminus F$$

because $\underline{E}_\mu(z^*)$ is path connected. This proves (6.41').

Hence (6.40) and (6.39) are shown too.

We finish now the proof (6.36). One of the minimal joins g_1, g_2 has the same end point as g_0 . Say $g_i, i \in \{1,2\}$ has the same end point as g_0 . By assumption (6.37) is $z_i \in a \setminus d_1$. Therefore by (6.41') we get

$$(6.43) \quad z_i \in E_2 \setminus F .$$

Using arguments like in the proof (6.38) we find that $g_i]0, \gamma_i] \cap S = \emptyset$. Clearly $g_i]0, \gamma_i] \cap d_2 = \emptyset$ and $g_i]0, \gamma_i] \cap d_1 = \emptyset$ by definition of γ_i and because $z_i \notin d_1$. Therefore

$$(6.43') \quad g_i]0, \gamma_i] \cap \partial F = \emptyset .$$

Thus using (6.43) and (J) we get

$$(6.44) \quad g_i]0, \gamma_i] \subset (E_2 \setminus F) .$$

Hence (6.39) and (6.44) imply that there are two (as point sets) distinct minimal joins from x_k to p , a contradiction. The case where q is end point of g_0 can be treated in the same way. It is obvious that the cases described by (6.37') can be handled using the same ideas as above in the proof of (6.37). This proves (6.36).

We proceed now with the proof of (6.35). Let F be the topological disc with $\partial F := S \cup d_1 \cup d_2$.

By (6.36) we know that $z \in \tilde{a}$, \tilde{a} a subarc of a , \tilde{a} joining z_1 and z_2 . Let us assume say $\phi(z_1) > \phi(z_2)$ then by (6.36) we get $\phi(z_1) > \phi(z) > \phi(z_2)$.

Therefore $z_1 \in a \setminus d_1$ and $z_2 \in d_1 \setminus \{z\}$. Thus by (6.39)

and (6.44) we get

$$(6.45) \quad g_2]0, \gamma_2[\subset F \setminus \partial F ,$$

$$(6.45') \quad g_1]0, \gamma_1[\subset (E_1 \setminus F) \cup \partial F .$$

For the proof of (6.35) we will argue by contradiction.

Therefore let us assume we have a sequence of generalized plicas $(v'_n) \notin S$, $v'_n \neq x_k$, $|v'_n| < \frac{1}{2}$ with $\lim v'_n = x_k$.

Then at least one of the following possibilities must hold:

$$(6.46) \quad \text{A subsequence of } (v'_n) \text{ is contained in } F.$$

$$(6.46') \quad \text{A subsequence of } (v'_n) \text{ is contained in } E_1 \setminus F .$$

Let us treat first case (6.46). Here an (equally denoted) subsequence of (v'_n) must be contained in $F \setminus \partial F$ because $v'_n \notin S \cup d_2$. Let e_{pn}, e_{qn} be the minimal joins going from the generalized plica v'_n to the points p, q respectively. By Lemma 6.3 we know that e_{pn}, e_{qn} converge against two distinct (normalized) minimal joins $e_p(t), e_q(t)$ going from x_k to p, q respectively. Using the arguments from the proof of (6.38) and (6.39) we find that there exists a positive number α such that

$$(6.47) \quad e_p]0, \alpha[\cup e_q]0, \alpha[\subset F \setminus \partial F .$$

One of the paths e_p, e_q has the same end point as g_1 ; here e_p has the same endpoint as g_1 . Therefore (6.45') and (6.47) imply that g_1, e_p are two distinct minimal joins going from x_k to p , a contradiction.

In case of (6.46') by similar arguments we get two distinct normalized minimal joins $\bar{e}_p(t), \bar{e}_q(t)$ going from x_k to p, q

respectively and we find a number $\bar{\alpha} > 0$ such that

$$(6.48) \quad \bar{e}_p[0, \bar{\alpha}] \subset (E_1 \setminus \partial F) \cup \partial F ,$$

$$(6.48') \quad \bar{e}_q[0, \bar{\alpha}] \subset (E_1 \setminus \partial F) \cup \partial F .$$

Hence (6.45) and (6.48') imply here that g_2, \bar{e}_q are two (as point sets) distinct minimal joins from x_k to q , a contradiction.

It is obvious that all other possible cases for the assumptions in the proof of (6.35) (i.e. where say e.g. $\phi(z_2) > \phi(z_1)$ and where say e.g. g_1 is a minimal join from x_k to q) can be treated using the same arguments as above. This proves (6.35). Thus we have shown (6.34).

Using the preceding considerations the proof for the remaining parts of theorem 6.3 is now fairly easy.

Proof of theorem 6.3 a₂): We want to show first that

$K = \Psi(I_{-1} \cup I_I)$ agrees with $C_{\{p,q\}} \setminus \partial M$.¹⁾ For this, let $y_0 \in M \setminus \partial M$ be any limit of picas in $C_{\{p,q\}}$. We have to show now that $y_0 \in K$. By Lemma 6.3, we know that y_0 must be a generalized pica. Therefore there exist exactly two normalized minimal joins $g_{y_0 p}(t), g_{y_0 q}(t)$ going from y_0 to p, q respectively; further

$$\text{length } g_{y_0 p} = \text{length } g_{y_0 q} = d(y_0, \{p, q\}) \text{ and } g_{y_0 p} \cap g_{y_0 q} = \{y_0\}.$$

Like in the proof of (6.31) let $\bar{g}(t)$ be the normalized minimal join from p to q . Let $b_1 := g_{y_0 p} \cap \bar{g}$, $b_2 := g_{y_0 q} \cap \bar{g}$ and

1) This means K is the only connected component of $C_{\{p,q\}} \setminus \partial M$.

define \tilde{b} to be the closure of $\bar{g} \setminus (b_1 \cup b_2)$. Clearly like in (6.31) the point m is an interior point of \tilde{b} ; and $\tilde{b}^* := (g_{y_0 p} \setminus b_1) \cup (g_{y_0 q} \setminus b_2) \cup \tilde{b}$ is boundary of a topological disc \tilde{B} , $\tilde{B} \subset M$. Using the arguments in the proofs of (6.31), (6.32), (6.32') it is easily seen that there exists a number $l \in \{-1, 1\}$ such that an initial piece of $\Psi(I_1 \setminus \{0\})$ is contained in $\tilde{B} \setminus \partial\tilde{B}$ and $\Psi(I_1) \cap (\partial\tilde{B} \setminus \{m\}) = \{y_0\}$. Thus $y_0 \in K$. This shows $\Psi(I_{-1} \cup I_1) = C_{\{p, q\}} \setminus \partial M$.

Now we prove:

$$(6.49) \quad C_{\{p, q\}} = \overline{\Psi(I_{-1})} \cup \overline{\Psi(I_1)}.$$

Clearly $C_{\{p, q\}} \supset \overline{\Psi(I_{-1})} \cup \overline{\Psi(I_1)}$ because every point $y \in \Psi(I_{-1} \cup I_1)$ is a pica relative to $\{p, q\}$. It remains to show:

$$(6.49') \quad C_{\{p, q\}} \subset \overline{\Psi(I_{-1})} \cup \overline{\Psi(I_1)}.$$

For this let $\bar{y}_0 \in C_{\{p, q\}}$. We must prove that $\bar{y}_0 \in \overline{\Psi(I_{-1})} \cup \overline{\Psi(I_1)}$. Obviously, because of $\Psi(I_{-1} \cup I_1) = C_{\{p, q\}} \setminus \partial M$ we can assume that $\bar{y}_0 \in \partial M$. Thus the boundary point \bar{y}_0 is a limit of picas.¹⁾

Therefore using the proofs of (6.31), (6.32), (6.33), (6.35) it is easy to see that there exists a (unique) number $l' \in \{-1, 1\}$ such that $\bar{y}_0 = \lim \Psi(t_n)$, (t_n) an arbitrary sequence in I_1 , with $\lim t_n = l'$. Thus $\bar{y}_0 \in \overline{\Psi(I_1)}$.

This proves (6.49).

1) We can also assume that \bar{y}_0 is limit of generalized picas, it makes no difference here!

Finally, using the considerations for the proof of (6.35) together with the fact that Ψ is an embedding, it is obvious that $\overline{\Psi(I_{-1})} \cap \overline{\Psi(I_1)} = \{m\}$. Hence the proof of theorem 6.3a₂) is complete.

Proof of theorem 6.3 b₁), b₂)

We know by proposition 6.2 that there exists

in $M \setminus \partial M$ a pica relative to $\{p, q\}$.

Hence we can choose $x' \in C_{\{p, q\}} \cap (M \setminus \partial M)$.

By proposition 6.3c) we know that x' is a

pica relative to $\{p, q\}$. Let $g_{x', p}, g_{x', q}$ be the minimal

joins from x' to p, q respectively. Let \bar{g} be the minimal

join from p to q . We define $\bar{b}_1 := g_{x', p} \cap \bar{g}$, $\bar{b}_2 := g_{x', q} \cap \bar{g}$

and we denote with \hat{b} the closure of $\bar{g} \setminus (\bar{b}_1 \cup \bar{b}_2)$. Clearly like

in the proof of (6.31), (6.32) the curve $\hat{b} := (g_{x', p} \setminus b_1) \cup (g_{x', q} \setminus b_2) \cup \hat{b}$

is boundary of a topological disc B' , $B' \subset M$. By propo-

sition 6.3b) exists a number $\varepsilon' > 0$ and a C^1 -smooth embedding

$\tilde{\Psi} :]t' - \varepsilon, t' + \varepsilon[\rightarrow M \setminus \partial M$ with $t' = \frac{1}{2}$, $\varepsilon = \frac{1}{4}$, $\Psi(t') = x'$

such that

$$\tilde{\Psi}]t' - \varepsilon, t' + \varepsilon[= A_{\{p, q\}} \cap \overset{\circ}{B}_{\varepsilon'}(x') = C_{\{p, q\}} \cap \overset{\circ}{B}_{\varepsilon'}(x'),$$

$\overset{\circ}{B}_{\varepsilon'}(x') = \{y \in M / d(x', y) < \varepsilon'\}$. Using proposition 6.3c) we

know ¹⁾ that there exists a number $\bar{\varepsilon} > 0$ such that

$\tilde{\Psi}]t' - \bar{\varepsilon}, t' [\subset B' \setminus \partial B'$. There exists a connected component K' of $C_{\{p, q\}} \setminus M$ with $\tilde{\Psi}]t' - \varepsilon, t' + \varepsilon[\subset K'$.

We know by proposition 6.3d) that K' is the image of a

C^1 -smooth embedding of the intervall $]0, 1[$. Clearly we can

1) In case of need we change the orientation of $\tilde{\Psi}$.

assume that $\tilde{\Psi}:]t' - \varepsilon, t' + \varepsilon[\rightarrow M \setminus \partial M$ is a restriction of this (here equally denoted) embedding $\tilde{\Psi}:]0, 1[\rightarrow M \setminus \partial M$, with $\tilde{\Psi}(]0, 1[) = K'$. Using (J) we find that there are now two possibilities:

$$(6.50) \quad \Psi(]0, t'[) \cap \partial B' \neq \emptyset,$$

$$(6.50') \quad \tilde{\Psi}(]0, t'[) \subset B' \setminus \partial B'.$$

In case of (6.50) the considerations for the proof of (6.32') imply $\tilde{\Psi}(]0, t'[) \cap \partial B' = \{m\}$, a contradiction because $m \in \partial M$. Therefore (6.50') holds. Using proposition 6.3e) (cf. also (6.32), (6.33)) it is easy to see that the set $\tilde{\Psi}(]0, t'[)$ has a cluster point $x^* \in \partial B' \cap \partial M$. Thus by (6.32') we get $x^* = m$.

A consequence of what we have shown up to now in our proof of theorem 6.3b) can be stated as follows:

$$(6.51) \quad \text{"If } \bar{K} \text{ is an arbitrary connected component of } C_{\{p, q\}} \setminus \partial M \text{ then } m \in \partial M \text{ is a cluster point of } \bar{K} \text{"}$$

Combining (6.51) with the considerations in the proof of (6.34), (6.35) we find that $\bar{K} = K' = \tilde{\Psi}(]0, 1[)$. Thus

$$(6.52) \quad K' = \tilde{\Psi}(]0, 1[) = C_{\{p, q\}} \setminus \partial M.$$

By the preceding arguments and proposition 6.3e) it is now obvious that for every sequence (t_n) in $]0, 1[$ with $\lim t_n = 0$ the sequence $\tilde{\Psi}(t_n)$ must converge to the point m . Thus defining $\tilde{\Psi}(0) = m$, we can extend the map $\tilde{\Psi}$, i.e.:

$$(6.53) \quad \text{we have a continuous embedding } \tilde{\Psi}:]0, 1[\rightarrow M.$$

Clearly (6.52) and (6.53) make the proof of theorem 6.3b₁) complete. Theorem 6.3b₂) is an immediate consequence of the corresponding results in theorem 6.3a₁) and theorem 6.3a₂).

Proof of theorem 6.3c):

Let the point $w \in M$ be a generalized pica relative to $\{p, q\}$. We shall prove that w is a cluster point of picas relative to $\{p, q\}$. Clearly if $w \in M \setminus \partial M$ then w is a pica and belongs to $C_{\{p, q\}}$. Therefore let us assume $w \in \partial M$. We know by proposition 6.2 that there exists in $M \setminus \partial M$ a point \bar{w} being a pica relative to $\{p, q\}$. We use now considerations similar to those we have already applied several times e.g. the last time in the proof of (6.51). Therefore our proof will be very scetchy. Let g_{wp} , g_{wq} be the normalized minimal joins from w to p, q respectively. Denote with g_{wp}^-, g_{wq}^- the (normalized) minimal joins from \bar{w} to p, q respectively. Using arguments like in the proof of 6.31 we find that $g_{wp}^- \cap g_{wq}^- = \emptyset$, $g_{wq}^- \cap g_{wp}^- = \emptyset$. Let $h_1 := g_{wp} \cap g_{wp}^-$, $h_2 := g_{wq}^- \cap g_{wq}$ and define h' to be the closure of $(g_{wp}^- \setminus h_1) \cup (g_{wq}^- \setminus h_2)$. Like on page 191 the curve $\tilde{h} := (g_{wp} \setminus h_1) \cup h' \cup (g_{wq} \setminus h_2)$ is boundary of a topological disc W . We know by theorem 6.3a₂) and theorem 6.3b₁) that there exists a C^1 -smooth embedding $\Psi^* := I_{-1} \cup I_1 \rightarrow M$ with $\Psi^*(0) = \bar{w}$ and $\Psi^*(I_{-1} \cup I_1) = C_{\{p, q\}} \setminus \partial M$. Using the arguments in the proof of (6.32), (6.33) we find that there exists a number $k' \in \{-1, 1\}$ such that $\Psi^*(I_{k'}) \subset (W \setminus \partial W) \setminus \{\bar{w}\}$ and w is a cluster point of

$\Psi^*(I_k)$. Hence w is a limit of picas because all points in $\Psi^*(I_k)$ are picas. This proves theorem 6.3 c).

Proof of theorem 6.3d): In case $m \notin \partial M$ the claim of theorem 6.3d) is a consequence of proposition 6.3b). Let us therefore assume that $m \in \partial M$. We argue by contradiction. Namely assume there exists a sequence (\tilde{v}_n) in $A(p,q)$ with $\lim \tilde{v}_n = m$ such that the following holds:

"There exist normalized minimal joins g_{qn}, g_{pn} from \tilde{v}_n to p, q respectively and we have positive numbers ε_n such that $g_{qn} [0, \varepsilon_n] = g_{pn} [0, \varepsilon_n]$."

Let $\bar{\varepsilon}_n := \max \{ \varepsilon > 0 / g_{qn} [0, \varepsilon] = g_{pn} [0, \varepsilon] \}$.

Clearly like in the proof Lemma 6.3 (see p.) the point $g_{qn}(\bar{\varepsilon}_n) = g_{pn}(\bar{\varepsilon}_n)$ is a generalized pica relative to $\{p, q\}$ and $g_{pn}(\bar{\varepsilon}_n) \in \partial M$. Thus by theorem 6.3 the point $g_{pn}(\bar{\varepsilon}_n) \in C_{\{p, q\}} \cap \partial M$. Let $g_{mp}(t), g_{mq}(t)$ be the normalized minimal joins from m to p, q respectively. Since the sequences g_{qn}, g_{pn} converge against g_{mq}, g_{mp} respectively the sequence $g_{pn}(\bar{\varepsilon}_n)$ must converge to m . By (6.34) there exists an open neighbourhood $U(m)$ in M of the point m such that

$U(m) \cap C_{\{p, q\}} \subset \tilde{\Psi}([0, \frac{1}{2}[) ; \tilde{\Psi} : [0, 1[\rightarrow \partial M$ an embedding with $\tilde{\Psi}(0) = m, \tilde{\Psi}([0, 1[) = C_{\{p, q\}} \setminus \partial M$, cf.: theorem 6.3b₁).

Therefore there exists a number n_0 such that for all $n \in \mathbb{N}$ with $n > n_0$ is $g_{pn}(\bar{\varepsilon}_n) \in U(m) \cap C_{\{p, q\}} \subset \tilde{\Psi}([0, \frac{1}{2}[)$. This yields $g_{pn}(\bar{\varepsilon}_n) = m$ for all $n > n_0$ because

$(\tilde{\Psi}([0, \frac{1}{2}[) \cap \partial M = \{m\}$. Therefore for all $n > n_0$ we have

$$d(\tilde{v}_n, q) = d(\tilde{v}_n, m) + d(m, q),$$

$$d(\tilde{v}_n, p) = d(\tilde{v}_n, m) + d(m, p),$$

$$d(p, q) = d(p, m) + d(m, q) .$$

This yields a contradiction against remark 6.6. Therefore $\bar{\epsilon}_n = 0$ for all $n > n_0$. It is now easy to see that the preceding considerations prove theorem 6.3d).

With the notations and assumptions of theorem 6.3, proposition 6.3a and theorem 6.3 yield obviously the following

Corollary 6.4: For every point $x \in C_{\{p,q\}} \setminus \partial M$ there exists a number $t \in]-1, 1[$, ($t \in]0, 1[$) and $\frac{d\psi(t)}{dt}$, $(\frac{d\tilde{\psi}(t)}{dt})$ bisects the angle built by the two initial vectors of those two minimal joins going from the point x to the points p and q .

In the rest of this paragraph we shall be less pedantic than in the preceding part of this paragraph. We shall not always derive a claim in all details, when we believe that the validity of a claim is sufficiently clear.

Remark 6.9: Let S be a simply connected space of type (**). Then at least for a large class of examples S must be homeomorphic to $\mathbb{D} \cup T$ ¹⁾, $\mathbb{D} := \{x \in \mathbb{R}^2 / |x| < 1\}$, T an open subset of $S^1 := \{x \in \mathbb{R}^2 / |x| = 1\}$, $|\cdot|$ the Euclidean norm. Further it is plausible that for any two distinct points p, q the cut locus $C_{\{p,q\}}$ separates S . Namely using (J), theorem 6.3 and considerations like in the proof of Lemma 6.3 then at least in case S is compact it is clear that for any two distinct points p, q in S the set $S \setminus C_{\{p,q\}}$ has two components K_p, K_q ²⁾, $p \in K_p, q \in K_q$. Therefore every path $c : [0,1] \rightarrow S$ with $c(0) = p, c(1) = q$ meets $C_{\{p,q\}}$ i.e. there exists a number $t_0 \in]0,1[$ with $c(t_0) \in C_{\{p,q\}}$. Both components K_p, K_q are simply connected. ²⁾

1) In case S is compact this claim is a consequence of (J). In any case $S \setminus \partial S$ must be homeomorphic to the open unit disc. Namely $S \setminus \partial S$ is simply connected because $S \setminus \partial S$ has the same homotopy type as S . The latter holds because $S \setminus \partial S$ is a weak deformation retract of S , see [65] p. 297. Therefore say e.g. by the Riemannian mapping theorem $S \setminus \partial S$ must be homeomorphic to the open two-dimensional unit disc.

2) This holds also, because in case S is compact it can be shown that there exists a homeomorphism $h: S \rightarrow D, D := \{x \in \mathbb{R}^2 / |x| \leq 1\}$, and $h(C_{\{p,q\}})$ is a diameter of D , see [21], p. 141.

Using theorem 6.3 we prove now a result which describes the cut locus of a point in certain surfaces having the homotopy type of an annulus.

Theorem 6.5: Let S be a closed bordered subsurface of a space M . We assume that M is a two-dimensional, complete, unbordered, simply connected Riemannian manifold with curvature everywhere smaller than or equal to zero. ¹⁾

We assume further that S has locally rectifiable boundary curves and that:

$$\begin{aligned} (*)' \quad S \text{ is homeomorphic to } \bar{A} &:= \overset{*}{T}_1 \cup \overset{*}{A} \cup \overset{*}{T}_2 \quad 2), \\ \overset{*}{A} &:= \{x \in \mathbb{R}^2 \mid 1 < |x| < 2\}, \quad \overset{*}{T}_1 := \{x \in \mathbb{R}^2 \mid |x| = 1\}, \\ \overset{*}{T}_2 &\text{ an open subset of } \{x \in \mathbb{R}^2 \mid |x| = 2\}, \\ &\quad | \text{ the Euclidean norm } \end{aligned}$$

Let p be any point in S . Then the following holds:

-
- 1) In other words M is diffeomorphic to the Euclidean plane, M is complete and the curvature is on M nowhere positive.
 - 2) Condition $(*)$ says in short that S is some kind of generalized annulus. Note we can replace condition $(*)$ by the weaker one saying that S has the homotopy type of a circle! We use condition $(*)$ only for technical reasons to simplify the description later on! We can replace condition $(*)$ also by saying: "If S is compact then S has exactly two boundary components. If S is non-compact then ∂S has exactly one compact component." It is well known that under these assumptions $M \setminus \partial S$ has exactly one bounded component and this component is homeomorphic to the open unit disc.

- a) The cut locus C_p is homeomorphic to one of the two following intervals $[0,1]$ or $[0,1[$.

We have a continuous embedding

$$\bar{\Psi} : J \rightarrow S, \quad J \in \{[0,1], [0,1[\},$$

and

$$\bar{\Psi}(J) = C_p.$$

The restriction

$$\bar{\Psi}' :]0,1[\rightarrow S$$

is a C^1 -smooth embedding and

$$\bar{\Psi}'(]0,1[) = C_p \setminus \partial S.$$

- b) We have

$$\bar{\Psi}(0) \in T_1,$$

T_1 being the frontier and boundary of the bounded component of $M \setminus S$.

If $1 \in J$ i.e. C_p is homeomorphic to $[0,1[$, then C_p is unbounded. Otherwise $\bar{\Psi}(1) \in \partial S \setminus T_1$.

- c) A point q belongs to C_p if q is a generalized pica¹⁾ relative to p .

For every point $q \in C_p$ there exist exactly two distinct (normalized) minimal joins $c_1[0,d(p,q)]$, $c_2[0,d(p,q)]$ going from p to q and there exists a number $\delta \in [0,d(p,q)]$ such that

1) Cf. definition 6.5.

$$c_1[0, \delta] = c_2[0, \delta]$$

and

$$c_1[\delta, d(p, q)] \cup c_2[\delta, d(p, q)]$$

yields a non-trivial simple loop.¹⁾

The loop

$$c_1[0, d(p, q)] \cup c_2[0, d(p, q)]$$

is homotopic to the (modulo orientation) unique shortest non-trivial loop with base point p .

We have $m \in C_p$ and $d(p, m) = d(p, C_p)$, m being the mid point of the (unique) shortest non-trivial loop with base point p .

- d) Let $q = \bar{\psi}(t) \in C_p \setminus \partial S$. Then $\frac{d\bar{\psi}(t)}{dt}$ bisects the angle built by the two initial vectors of those two minimal joins going from the point q to the point p .
- e) For every pair of points $w_1, w_2 \in S$ there exist at most two distinct normalized minimal joins, going from w_1 to w_2 .

Proof of theorem 6.4:

We shall make use of the following assertion:

(6.54) Let r be any point in $\bar{A} \subset \mathbb{R}^2$ and let

$$g_1, g_2 : [0, 1] \rightarrow \bar{A},$$

1) The number δ is depending on p and q .

$$g_1(0) = g_2(0) = g_1(1) = g_2(1) = r$$

be any two non trivial simple loops in \bar{A} with base point r . Then modulo the orientation g_1 and g_2 are homotopic in \bar{A} .

Assertion (5.54) holds because the winding number of g_1, g_2 with respect to any point $r_0 \in \overset{\circ}{D} := \{x \in \mathbb{R}^2 / |x| < 1\}$ is contained in $\{1, -1\}$, cf. remark 6.2. It is well known that in the above space of type $\bar{A} \subset \mathbb{R}^2$ the winding number with respect to any point in $\overset{\circ}{D}$ determines the homotopy class of a closed path in \bar{A} . This proves (6.54).

We shall also use the subsequent assertion:

(6.55) Let p_1, p_2 be any two points in S .

Let $\alpha_1, \alpha_2, \alpha_3 : [0, a] \rightarrow S$ be three normalized minimal joins from p_1 to p_2 , then at least two of them must be homotopic.

This assertion is intuitively very clear because a minimal join cannot wrap around the boundary curve T_1 .

Therefore we think that we can omit this proof. A detailed proof of (6.55) can be given by using a rather lengthy and tedious discussion of the possible positions for all subloops of $\alpha_1 [0, a] \cup \alpha_2 [0, a] \cup \alpha_3 [0, a]$ in relation to the boundary curve T_1 .

Using (6.55) we prove now:

(6.56) In the above space S any two points can be joined by at most two as point sets distinct minimal joins. ¹⁾

Proof of (6.56): Let r_1, r_2 be any two (distinct) points in S . Assume there exist three distinct normalized minimal joins $\beta_1, \beta_2, \beta_3 : [0, \bar{a}] \rightarrow S$ going from r_1 to r_2 . Then by (6.55) two of them say β_1 and β_3 must be homotopic. Let (\tilde{S}, \tilde{d}) be the universal covering space of (S, d) , $\pi : \tilde{S} \rightarrow S$ the covering projection. We move along $\beta_1 [0, \bar{a}]$ from r_1 to r_2 , then we move back to r_1 along $\beta_3 [0, \bar{a}]$. We denote the just described path by $g : [0, 2\bar{a}] \rightarrow S$. We lift g to \tilde{S} and we start the lift in $\tilde{r}_1 \in \pi^{-1}(r_1)$. Denote this lift by $\tilde{g} : [0, 2\bar{a}] \rightarrow \tilde{S}$. We have $\tilde{g}(\bar{a}) \in \pi^{-1}(r_2)$, let $\tilde{r}_2 := \tilde{g}(\bar{a})$. Clearly $g[0, 2\bar{a}] \rightarrow S$ is nullhomotopic, because β_1, β_3 are homotopic. Therefore $\tilde{g}(2\bar{a}) = \tilde{r}_1$. Therefore theorem 6.2f) yields $g[0, \bar{a}] = g[\bar{a}, 2\bar{a}]$ because S is simply connected ²⁾. Thus $g[0, \bar{a}] = g[\bar{a}, 2\bar{a}]$. Hence $\beta_1[0, \bar{a}] = \beta_3 [0, \bar{a}]$, a contradiction. It obvious that the remaining cases where β_1 is homotopic to β_2 and β_2 is homotopic to β_3 can be treated in the same way. This proves (6.56).

After these preparations, we start now the proper proof of theorem 6.4. For this, let p be any point in (S, d) .

1) We shall use (6.56) later on several times.

2) cf. remark 6.10 i.e. footnote 2) on the next page.

There exists a shortest, non-contractable (normalized) loop

$\alpha : [0, 2L] \rightarrow S$ in S with base point $\alpha(0) = \alpha(2L) = p$ ¹⁾.

As above let (\tilde{S}, \tilde{d}) be the universal covering space of (S, d) and let $\tilde{\pi} : \tilde{S} \rightarrow S$ be the covering projection.

Lift the path α to \tilde{S} . We start the lift at some point $\tilde{p}_1 \in \tilde{\pi}^{-1}(p)$ and we denote the normalized lifted path

by $\tilde{\alpha} : [0, 2L] \rightarrow \tilde{S}$. Clearly $\tilde{p}_2 := \tilde{\alpha}(2L) \in \tilde{\pi}^{-1}(p) \setminus \{\tilde{p}_1\}$ and $2L = \tilde{d}(\tilde{p}_1, \tilde{p}_2)$ ¹⁾.

By Theorem 6.3 ²⁾ $\tilde{m} := \tilde{\alpha}(L) \in C_{\{\tilde{p}_1, \tilde{p}_2\}}$ and $m := \pi(\tilde{m}) \in C_p$ ¹⁾.

There are now two possibilities:

$$(6.57) \quad \tilde{m} \notin \partial\tilde{S}$$

$$(6.58) \quad \tilde{m} \in \partial\tilde{S}$$

We treat first case (6.57). By theorem 6.3 ²⁾ there exists a continuous embedding

$$\Psi : J_{-1} \cup J_1 \rightarrow \tilde{S}$$

$$J_{-1} \in \{[-1, 0], [1, 0]\}, \quad J_1 \in \{[0, 1], [0, 1[[]\}, \quad \text{³⁾$$

1) cf. the proof of Lemma 6.2.

2) Remark 6.10: In order to apply theorem 6.3, 6.2 we use here that (\tilde{S}, \tilde{d}) can be viewed as a subspace of a simply connected, complete, unbordered, two-dimensional Riemannian manifold \tilde{M} without conjugate points. Now we explain why this is possible. S is a subspace of M , M being diffeomorphic to R^2 and the curvature on M is everywhere smaller than or equal to zero. The path T_1 is boundary of a topological disc B contained in M . We delete a point $p_0 \in (B \setminus T_1)$ from the space M . (Continued next page!)

3) Note this definition of $J_{-1} \cup J_1$ is different from that one used in theorem 6.3!!

such that $\Psi(J_{-1} \cup J_1) = C_{\{\tilde{p}_1, \tilde{p}_2\}}$, $\Psi(O) = \tilde{m}$. The restriction

$\Psi/ : J_{-1}^* \cup J_1^* \rightarrow \tilde{S}$, $J_{-1}^* := J_{-1}^* \setminus \{-1\}$, $J_1^* := J_1 \setminus \{1\}$ is a C^1 -smooth embedding and $\Psi(J_{-1}^* \cup J_1^*) = C_{\{\tilde{p}_1, \tilde{p}_2\}} \setminus \delta \tilde{S}$.

Now we claim:

$$(6.59) \quad \pi(C_{\{\tilde{p}_1, \tilde{p}_2\}}) = C_p.$$

(6.60) A point q is in C_p iff q is a generalized pica¹⁾ relative to p . For every point $q \in C_p$ there exist exactly two distinct (normalized) minimal joins $c_1[0, d(p, q)]$, $c_2[0, d(p, q)]$ going from p to q and there exists a number $\delta^2) \in [0, d(p, q)]$ such that $c_1[0, \delta] = c_2[0, \delta]$ and $c_1[\delta, d(p, q)] \cup c_2[\delta, d(p, q)]$ yields a non-trivial simple loop. The loop $c_1[0, d(p, q)] \cup c_2[0, d(p, q)]$ is homotopic to the (modulo orientation) unique shortest, non-trivial loop with base point p . We have $m = \pi(\tilde{m}) \in C_p$ and $d(p, C_p) = d(p, m)$, m the mid point of the unique shortest non-trivial loop with base point p .

cont. footnote 2) (Remark 6.10) from p. :

We change the metric on $M \setminus \{p_0\}$ in a small geodesic disc $B_\delta(p_0) \setminus \{p_0\} \subset B \setminus T^1$ such that the new space \tilde{M} is complete and has¹⁾ curvature everywhere smaller than or equal to zero. Such a change of the metric on $M \setminus \{p_0\}$ is possible. This is intuitively clear and can be shown precisely by a straightforward construction on the punctured geodesic disc $B_\delta(p_0) \setminus \{p_0\}$. Clearly this change of the metric and topology²⁾ on M has no effect on the intrinsic structure of (S, d) . Now S is a deformation retract of M . (cont. next page!)

1) cf. definition 6.5.

2) δ is depending on p and q .

$$(6.61) \quad \pi \circ \Psi : J_{-1} \cup J_1 \rightarrow S$$

is an embedding, the restriction

$$\pi \circ \Psi : J_{-1}^* \cup J_1^* \rightarrow S \text{ is a } C^1\text{-smooth embedding,}$$

$$\pi \circ \Psi(J_{-1}^* \cup J_1) = C_p \setminus \partial S .$$

$$(6.61') \quad \text{Let } q = \pi \circ \Psi(t) \in C_p \setminus \partial S \text{ then } \frac{d\pi \circ \Psi(t)}{dt}$$

bisects the angle built by the two initial

vectors of those two minimal jons going from q

to the point p .

$$(6.62) \quad \text{If } 1 \notin J_{-1} \cup J_1, \quad (-1 \notin J_{-1} \cup J_1) \text{ then}$$

$d(\pi \circ \Psi(t_n), p)$ is an unbounded sequence for

every sequence (t_n) in $J_{-1} \cup J_1$ with

$$\lim t_n = 1 \quad (\lim t_n = -1).$$

$$(6.63) \quad J_{-1} = [-1, 0].$$

$$(6.64) \quad \pi \circ \Psi(-1) \in T_1. \text{ If } 1 \in J_{-1} \cup J_1 \text{ then}$$

$$\pi \circ \Psi(1) \in (\partial S \setminus T_1).$$

In order to prove (6.59) we show first

$$(6.59a) \quad \pi(C_{\{\tilde{p}_1, \tilde{p}_2\}}) \subset C_p .$$

cont. footnote 2) (Remark 6.10) from p. :

Therefore the embedding $i : S \rightarrow \bar{M}$ induces an isomorphism $i_* : \pi_1(S) \rightarrow \pi_1(\bar{M})$, $\pi_1(S)$, $\pi_1(\bar{M})$ the fundamental groups of S, \bar{M} respectively. Therefore the universal covering space (\tilde{S}, \tilde{d}) can be viewed as a subspace of the universal covering space \tilde{M} of \bar{M} , see [50]p. 179, example 11.5. Clearly \tilde{M} is a simply connected, complete, two dimensional, unbordered Riemannian manifold without conjugate points. Hence $\tilde{M} \supset \tilde{S}$ has all properties wanted above.

For this we proceed as follows. Let

$$G := \{ \tilde{x} \in \Psi(J_{-1} \cup J_1) / \pi(\tilde{x}) \in C_p \}.$$

Now (6.59a) is true, if we can prove that G is a non empty, open and closed subset of $\Psi(J_{-1} \cup J_1) = C_{\{\tilde{p}_1, \tilde{p}_2\}}$.

Clearly $G \neq \emptyset$ because $\pi(\Psi(O)) = m \in C_p$ and G is closed because $\Psi(J_{-1} \cup J_1)$ ¹⁾, C_p are closed and because π is a local homeomorphism. It remains to prove that

$$(6.59b) \quad G \text{ is open in } \Psi(J_{-1}^* \cup J_1^*)$$

For this we show:

$$(6.59c) \quad \text{Let } t_o \in J_{-1}^* \cup J_1^* \text{ with } \pi(\Psi(t_o)) \in C_p \\ \text{then there exists } \epsilon > 0 \text{ such that} \\ \pi(\Psi([t_o - \epsilon, t_o + \epsilon])) \subset C_p.$$

Proof of (6.59c): Assume (6.59c) is not true. Then there exists a sequence t_n with $\lim t_n = t_o$ and $q_n := \pi(\Psi(t_n)) \notin C_p$ for all $n \in \mathbb{N}$. Therefore

$$L_n := d(q_n, p) < d(\tilde{p}_1, \Psi(t_n)).$$

Let $e_n [0, L_n]$ be a normalized minimal join from q_n to p .

Now lift $e_n [0, L_n]$ to \tilde{S} , start the lift in $\tilde{q}_n := \Psi(t_n)$ and denote the lifted paths by $\tilde{e}_n [0, L_n]$. Then we have

$$\tilde{e}_n(L_n) \in \pi^{-1}(p) \setminus \{\tilde{p}_1, \tilde{p}_2\}$$

because

$$\tilde{d}(\Psi(t_n), \tilde{e}_n(L_n)) = L_n < d(\Psi(t_n), \tilde{p}_1) = d(\Psi(t_n), \tilde{p}_2).$$

1) Note all points in $\Psi(J_{-1} \cup J_1)$ are generalized picas relative to $\{p_1, p_2\}$.

Now

$$(\pi^{-1}(p) \setminus \{\tilde{p}_1, \tilde{p}_2\}) \cap B_V(\Psi(t_0))$$

is finite, $v := 2 \tilde{d}(\Psi(t_0), \tilde{p}_1)$. Therefore $\tilde{e}_n(L_n)$ has a cluster point \tilde{p}_3 , $\tilde{p}_3 \in \pi^{-1}(p) \setminus \{\tilde{p}_1, \tilde{p}_2\}$

and $\tilde{e}_n(L_n)$ yields an (equally denoted) constant subsequence $\tilde{e}_n(L_n) = \tilde{p}_3$. Therefore there exists

$\lim \tilde{e}_n[0, L_n] = \tilde{e}[0, L_0]$ ¹⁾, $\tilde{e}[0, L_0]$ being the normalized minimal join from $\tilde{q}_0 := \Psi(t_0)$ to \tilde{p}_3 ,

$$\lim L_n = L_0 = \tilde{d}(\tilde{q}_0, \tilde{p}_3) = \tilde{d}(\tilde{q}_0, \tilde{p}_1) = \tilde{d}(\tilde{q}_0, \tilde{p}_2) = d(q_0, p),$$

$q_0 = \pi(\tilde{q}_0)$. Now let $\tilde{e}_1[0, L_0]$, $\tilde{e}_2[0, L_0]$ be the minimal joins from \tilde{q}_0 to \tilde{p}_1 , \tilde{p}_2 respectively. Then

$$\pi(\tilde{e}_1[0, L_0]), \pi(\tilde{e}_2[0, L_0]), \pi(\tilde{e}[0, L_0])$$

are three non-homotopic, hence distinct minimal joins from q_0 to p , a contradiction against (6.56). This proves (6.59c) and completes the proof (6.59a).

It remains to complete the proof of (6.59). We have to show that

$$\pi(\Psi(J_{-1} \cup J_1)) = C_p.$$

Now it is easily seen that $\pi(C_{\{\tilde{p}_1, \tilde{p}_2\}})$ is closed because we know now²⁾ that for all $t \in J_{-1} \cup J_1$ we have

$d(p, \pi \circ \Psi(t)) = \tilde{d}(\tilde{p}_1, \Psi(t))$. Therefore it is sufficient to prove that

1) If necessary we take here for $\tilde{e}_n[0, L_n]$ an equally denoted subsequence.

2) By the proof of (6.59a).

(6.59d) in S every (generalized) pica q_p relative to p is contained in $\pi(\Psi(J_{-1} \cup J_1))$.

Proof of (6.59d): Now we prepare the proof (6.59d).

If q_p is a (generalized) pica relative to p , then we have at least two distinct normalized minimal joins $c_1[0, L']$, $c_2[0, L']$ from p to q_p and there exists $\varepsilon > 0$ such that

$$c_1]L' - \varepsilon, L'[\cap c_2]L' - \varepsilon, L'[= \emptyset .$$

We show that

$$(6.65) \quad c_1[0, L'] \cup c_2[0, L']$$

yields a loop h with base point p and h is homotopic to a non-trivial simple loop.

For the proof of (6.65) let

$$\bar{\varepsilon} := \max \{ \varepsilon > 0 / c_1(]L' - \varepsilon, L'[) \cap c_2(]L' - \varepsilon, L'[) = \emptyset \} .$$

Clearly the point set

$$d_1 := c_1[L' - \bar{\varepsilon}, L'] \cup c_2[L' - \bar{\varepsilon}, L']$$

yields an equally denoted simple loop with base point

$$p_{\bar{\varepsilon}} = c_1(L' - \bar{\varepsilon}) . \text{ We have that}$$

(6.65a) the loop d_1 is non-trivial.

Namely otherwise by theorem 6.2f) and the arguments used to prove (6.56) the lift of d_1 to \tilde{S} would collapse to a minimal path joining two points $\tilde{p}_{\bar{\varepsilon}}, \tilde{q}_p, \tilde{p}_{\bar{\varepsilon}} \in \pi^{-1}(p_{\bar{\varepsilon}}), \tilde{q}_p \in \pi^{-1}(q_p)$. This yields a contradiction because d_1 is

a simple loop. This proves (6.65a).

Further we have

$$(6.65b) \quad c_1 [0, \bar{\varepsilon}] = c_2 [0, \bar{\varepsilon}]$$

Namely otherwise $c_1 [0, L']$, $c_2 [0, L']$, $\bar{c}_2 [0, L']$ with $\bar{c}_2 [0, \bar{\varepsilon}] := c_1 [0, \bar{\varepsilon}]$, $\bar{c}_2 [\bar{\varepsilon}, L'] := \bar{c}_2 [\bar{\varepsilon}, L']$ yield three distinct minimal joins from p to q_p , a contradiction against (6.56). This proves (6.65b).

It is obvious that the combination of (6.65a) and (6.65b) proves (6.65). Now we complete the proof of (6.59d). Combining (6.65) with (6.54) we know that

$$(6.65c) \quad \text{the loop } c_1 [0, L'] \cup c_2 [0, L'] \text{ (with an appropriate orientation) is homotopic to } \alpha [0, 2L] \text{ } ^1).$$

Therefore if we lift $c_1 [0, L'] \cup c_2 [0, L']$ to \tilde{S} and start this lift in \tilde{p}_1 then the lifted path ends up in \tilde{p}_2 . Denote this lift of $c_1 [0, L'] \cup c_2 [0, L']$ by $\tilde{c}_1 [0, L'] \cup \tilde{c}_2 [0, L']$. We start the lift with $\tilde{c}_1 [0, L']$ ²⁾. Let $\tilde{q}_p := \tilde{c}_1 (L')$.

It is clear that in (\tilde{S}, \tilde{d}) $\tilde{c}_1 [0, L']$, $\tilde{c}_2 [0, L']$ yield minimal joins from \tilde{q}_p to \tilde{p}_1 , \tilde{p}_2 respectively. Therefore

$$\tilde{q}_p \in C_{\{\tilde{p}_1, \tilde{p}_2\}} \text{ because } q_p \text{ is a (generalized) pica. Hence } \pi(\tilde{q}_p) = q_p \in \pi(\Psi(J_{-1} \cup J_1)) = \pi(C_{\{\tilde{p}_1, \tilde{p}_2\}}).$$

This proves (6.59d).

1) Recall $\alpha [0, 2L]$ is the above shortest non-contractable (normalized) loop with base point p ; therefore by (6.65) $\alpha [0, 2L]$ is homotopic to a non-trivial simple loop with base point p .

2) We assume that we have here already the appropriate orientation, otherwise we start the lift with $c_2 [0, L']$.

Hence the proof of (6.59) is now complete!

We also need:

(6.65') If $\alpha_1[0,2L]$ is an arbitrary (normalized) shortest non-trivial loop with base point p , then

$$\alpha_1[0,2L] = \alpha[0,2L].$$

Proof of (6.65'): It is easily seen that $\alpha_1(L)$ is a generalized pica relative to p . Therefore by (6.65c) (with an appropriate orientation of $\alpha_1[0,2L]$) we can assume that $\alpha_1[0,2L]$ is homotopic to $\alpha[0,2L]$. Therefore we have in \tilde{S} lifts $\tilde{\alpha}_1[0,2L], \tilde{\alpha}[0,2L]$ of $\alpha_1[0,2L], \alpha[0,2L]$ respectively, with $\tilde{\alpha}_1[0,2L], \tilde{\alpha}[0,2L]$ being minimal joins from \tilde{p}_1 to \tilde{p}_2 . By theorem 6.2f is $\tilde{\alpha}_1[0,2L] = \tilde{\alpha}[0,2L]$. Hence

$$\alpha_1[0,2L] = \pi \circ \tilde{\alpha}_1[0,2L] = \pi \circ \tilde{\alpha}[0,2L] = \alpha[0,2L].$$

This proves (6.65').

Finally, for the proof of (6.60) we shall use that

(6.60') we have $m = \pi(\tilde{m}) \in C_p$ and $d(p, C_p) = d(p, m)$,
 m the mid point of the (modulo orientation unique),
 shortest, non-trivial loop with base point p .

Proof of 6.60': By the proof of (6.59c) $\tilde{d}(\tilde{p}, \tilde{z}) = \tilde{d}(\pi(\tilde{p}), \pi(\tilde{z}))$

for all $\tilde{z} \in C_{\{\tilde{p}_1, \tilde{p}_2\}}$. Hence by (6.59)

$\tilde{d}(\tilde{p}_1, C_{\{\tilde{p}_1, \tilde{p}_2\}}) = d(p, C_p)$. Therefore using theorem 6.3

we get $\tilde{d}(\tilde{p}_1, \tilde{m}) = \tilde{d}(\tilde{p}_1, C_{\{\tilde{p}_1, \tilde{p}_2\}}) = d(p, C_p) = d(p, m)$.

This together with (6.65') proves (6.60').

Now the proof of (6.60) is an immediate consequence of the preceding results. Namely by theorem 6.3 every point in $C_{\{\tilde{p}_1, \tilde{p}_2\}}$ is a generalized pica. Therefore by (6.59), (6.59d), (6.56), (6.65a), (6.65b), (6.65c), (6.65'), (6.60') the proof of (6.60) is complete.

We start the

Proof of (6.61): We have to prove:

(6.61a) The mapping $\pi \circ \Psi : J_{-1} \cup J_1 \rightarrow S \subset M$
is injective ¹⁾.

In order to prove (6.61a), we show first that

(6.61b) the restriction $\pi \circ \Psi_f :]-1, 1[\rightarrow S \setminus \partial S \subset M$
is injective.

Proof of (6.61b): We argue by contradiction. Assume there exist $t_1, t_2 \in]-1, 1[$, $t_1 < t_2$ with $\pi \circ \Psi(t_1) = \pi \circ \Psi(t_2)$. It is easily seen that the closed path $\pi \circ \Psi : [t_1, t_2] \rightarrow M$ contains a simple, closed subloop f ²⁾. Now f is in M boundary of a closed topological disc F .

We know that $p \notin f$ because $p \notin C_p$ for $d(p, C_p) = L > 0$ by (6.60). There are now two possibilities

$p \in F \setminus f$ or $p \in M \setminus F$.

Assume $p \in F \setminus f$. Then pick any point $x \in (M \setminus F) \cap S$ ³⁾.

1) (6.61a) is only a necessary condition for (6.61).

2) Recall the restriction $\Psi_f : [t_1, t_2] \rightarrow (S \setminus \partial S)$ is a C^1 -smooth embedding and π is a local diffeomorphism.

3) It is easily seen that $(M \setminus F) \cap S \neq \emptyset$, $(F \setminus f) \cap S \neq \emptyset$.

There exists in S a minimal join c_x going from p to x .
 Clearly by (J) c_x must pass through $f \in C_p \cap (S \setminus \partial S)$.
 However, it is obvious that c_x is not any longer a minimal
 join to p after passing through $C_p \setminus \partial S$, a contradiction.
 The analogue argument works if $p \in M \setminus F$. This proves
 (6.61b).

We proceed now with the proof of (6.61a). The interval
 $J_{-1} \cup J_1$ has a priori one of the following types:

- a) $] -1, 1[$, b) $[-1, 1[$, c) $] -1, 1]$ d) $[-1, 1]$.

It is obvious that $\pi \circ \Psi(1), \pi \circ \Psi(-1) \notin \pi \circ \Psi(]-1, 1[)$ ¹⁾
 because $\{\pi \circ \Psi(1), \pi \circ \Psi(-1)\} \subset \partial S$ ²⁾ and $\pi \circ \Psi(]-1, 1[) \subset S \setminus \partial S$.

Therefore in order to complete the proof of (6.61a) it
 remains to discuss in case d) the possibility
 $\pi \circ \Psi(-1) = \pi \circ \Psi(1) = y \in \partial S$. In this case, locally in a
 neighbourhood of the point y the topological situation
 looks as described in figure 6.1 below.

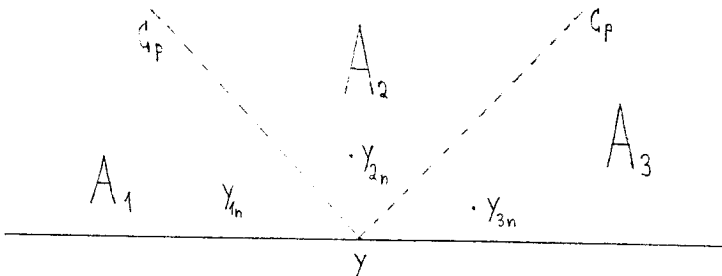


Figure 6.1

1) In case b), c) or d).

2) Recall by theorem 6.3a₁), if -1 , (or 1) $\in J_{-1} \cup J_1$
 then $\Psi(-1)$ (or $\Psi(1)$) $\in \partial S$.

This means the cut locus C_p ¹⁾ separates in S a small neighbourhood U of y in three disjoint regions A_1, A_2, A_3 ²⁾.

We can choose in every region a sequence of points

$$y_{1n} \in A_1, y_{2n} \in A_2, y_{3n} \in A_3 \quad \text{with} \quad \lim (y_{1n}) = \lim (y_{2n}) = \\ = \lim (y_{3n}) = y.$$

The initial pieces of minimal joins going from y_{1n}, y_{2n}, y_{3n} to p stay in A_1, A_2, A_3 respectively, because those minimal joins do not cross the cut locus C_p . The cluster points of those sequences of minimal joins yield at least three distinct minimal joins from y to p ²⁾, a contradiction against (6.56). This proves (6.61a).

We complete now the proof of (6.61) and we derive also (6.62). We showed in our proof of (6.59a) that for all $\Psi(t) \in \tilde{C}_{\{\tilde{p}_1, \tilde{p}_2\}}$ we have $\tilde{d}(\Psi(t), \tilde{p}_1) = d(\pi(\Psi(t)), p)$.

Therefore by theorem 6.3a₁) we get

$$(6.62) \quad \text{If } 1 \notin J_{-1} \cup J_1, \quad (-1 \in J_{-1} \cup J_1) \text{ then for} \\ \text{every sequence } t_n \text{ in } J_{-1} \cup J_1 \text{ with } \lim t_n = 1 \\ (\lim t_n = -1) \text{ the sequence } d(\pi \circ \Psi(t_n), p) \text{ is un-} \\ \text{bounded.}$$

Using (6.62) it is now easy to see that $\pi \circ \Psi : J_{-1} \cup J_1 \rightarrow S$ is proper. The latter together with (6.61a) and the con-

1) The cut locus C_p in figure 6.1 is dotted and ∂S is described by the horizontal, non-dotted line.

2) cf. the considerations in our proof theorem 6.3.

tinuity of $\pi \circ \Psi$ implies that $\pi \circ \Psi: J_{-1} \cup J_1 \rightarrow S$ is a continuous embedding. Further the facts that $\Psi: J_{-1}^* \cup J_1^* \rightarrow \tilde{S} \setminus \partial \tilde{S} \subset \tilde{\bar{M}}$ is a C^1 -smooth embedding ¹⁾ and that $\pi: \tilde{\bar{M}} \rightarrow \bar{M}$ is a local diffeomorphism imply that $\pi \circ \Psi: J_{-1}^* \cup J_1^* \rightarrow (S \setminus \partial S) \subset M$ is a C^1 -smooth embedding. Obviously $\pi \circ \Psi(J_{-1}^* \cup J_1^*) = C_p \setminus \partial S$ because $\Psi(J_{-1}^* \cup J_1^*) = C_{\{\tilde{p}_1, \tilde{p}_2\}} \setminus \partial \tilde{S}$, by theorem 6.3. Hence the proof of (6.61) is finished.

Using now corollary 6.4 we easily get (6.61'). This proves (6.61'). We prepare now the proofs of (6.63) and (6.64). For this let

$$\delta_1 := \max \{ \delta' \geq 0 / \alpha[L - \delta', L] \cap \alpha[L, L + \delta'] = \emptyset \},$$

$\alpha: [0, 2L] \rightarrow S$ the shortest non-trivial loop with base point p .

Clearly $\delta_1 > 0$ and

$$f_1 := \alpha[L - \delta_1, L] \cup \alpha[L, L + \delta_1]$$

is a simple subloop of $\alpha[0, 2L]$. Now f_1 is boundary of a closed topological disc F_1 contained in M . This topological disc F_1 is not contained in S because f_1 is non-trivial by (6.65a). It is easily seen that F_1 containing points of $M \setminus S$ contains the bounded component K_1 of $M \setminus S$. Clearly F_1 being closed also contains the frontier T_1 of K_1 . We prove now

$$(6.66) \quad F_1 \setminus f_1 \text{ contains no point } x \text{ belonging to } \partial S \setminus T_1.$$

1) This holds by theorem 6.3.

Proof of (6.66): We argue by contradiction. Assume there exists $x \in (F_1 \setminus f_1) \cap (\partial S \setminus T_1)$ and let K_x be the component of ∂S containing x . Clearly x is a point in the frontier of a component U_x of $M \setminus S$. There exists only one bounded component K_1 of $M \setminus S$. The frontier of K_1 is T_1 and $x \notin T_1$. Therefore U_x is unbounded. Hence U_x contains a point $u_x \notin F_1$. Since $x \in F_1 \setminus f_1$ is a cluster point of U_x there exists a point $x' \in U_x \cap (F_1 \setminus f_1)$. Since U_x is path connected there must exist a path $\eta \subset U_x$, η joining x' with u_x . However, by (J) $\eta \cap f_1 \neq \emptyset$, a contradiction because $\eta \subset U_x \subset M \setminus S \subset M \setminus f_1$ for $f_1 \subset \alpha[0, 2L] \subset S$. This proves (6.66).

Using from above that $T_1 \subset F_1 \cap \partial S$, (6.66) yields

$$(6.66a) \quad (F_1 \setminus f_1) \cap \partial S \subset T_1.$$

We shall make use of (6.66a) later on in the proof (6.64).

Next we derive some more statements for the proof of (6.63) and (6.64).

Using corollary 6.4 we know that for (some sufficiently small $\varepsilon > 0$) $\pi \circ \Psi[-\varepsilon, \varepsilon]$ is transversal to f_1 at $\pi \circ \Psi(0) = m$. Therefore with an appropriate orientation of Ψ there exists $\varepsilon > 0$ such that

$$(6.67) \quad \pi \circ \Psi]-\varepsilon, 0[\subset F_1 \setminus f_1$$

and $\pi \circ \Psi]0, \varepsilon[\subset M \setminus F_1$.

Next we prove that

$$(6.68) \quad f_1 \cap C_p = m = \pi \circ \Psi(0)$$

Assume the contrary. Then there exists a real number v , $0 < |v| < L$ and $\alpha(L + v) \in f_1 \cap C_p$. Clearly $L^* := d(\alpha(L + v), p) = L - |v| < L$.

By (6.60) we have two distinct minimal joins c_1, c_2 from p to q and $c_1 \cup c_2$ yields a non-trivial loop with base point p and length $2L^* < 2L$. This gives a contradiction because the shortest non-trivial loop with base point p has length $2L$. This proves (6.68).

Now we finish the proof of (6.63) and (6.64). Combining (6.67), (6.68), (6.61)¹⁾ and (J) we find

$$(6.69a) \quad \pi \circ \Psi(J_{-1} \setminus \{0\}) \subset F_1 \setminus f_1$$

$$(6.69b) \quad \pi \circ \Psi(J_1 \setminus \{0\}) \subset M \setminus F_1.$$

Hence combining (6.69a) and (6.62) we get $-1 \in J_{-1}$.

This proves (6.63).

By theorem 6.3 $\Psi(-1) \in \partial \tilde{S}$. Thus using (6.69a)

$$\pi \circ \Psi(-1) \in \partial S \cap (F_1 \setminus f_1).$$

Therefore using (6.66a) we get $\pi \circ \Psi(-1) \in T_1$.

This proves the first part of (6.64).

If $1 \in J_{-1} \cup J_1$ then by theorem 6.3 $\Psi(1) \in \partial \tilde{S}$. Hence $\pi \circ \Psi(1) \in \partial S$. Therefore combining the fact that $T_1 \subset F_1$ with (6.69b) we easily get the implication:

1) Here we use from (6.61) that $\pi \circ \Psi: J_{-1} \cup J_1 \rightarrow M$ is an embedding.

"If $1 \in J_{-1} \cup J_1$ then $\pi \circ \Psi(1) \in \partial S \setminus T_1$."

Hence the proof of (6.64) is complete.

Clearly (6.61') is now an immediate consequence of corollary 6.4. Therefore now all our claims (6.59), (6.60), (6.61), (6.61'), (6.62), (6.63), (6.64) are proved.

It remains to treat case

$$(6.57) \quad \tilde{m} = \tilde{\alpha}(L) \in \partial \tilde{S}.$$

In this case we get with some minor obvious modifications (based on the modifications in the corresponding part of theorem 6.3) the same statements as in case (6.56).

It is easily seen that in case (6.57) all these statements can be proved using the same methods as in case (6.56). Therefore we omit the proofs and give only the subsequent results.

By theorem 6.3 there exists a continuous embedding

$$\Psi: J_1 \rightarrow \tilde{S}$$

$$J_1 \in \{ [0,1], [0,1[]$$

$$\text{such that } \Psi(J_1) = C_{\{\tilde{p}_1, \tilde{p}_2\}}, \quad \Psi(0) = \tilde{m} \in \partial \tilde{S}.$$

The restriction

$$\Psi, :]0,1[\rightarrow \tilde{S}$$

is a C^1 -smooth embedding and

$$\Psi(]0,1[) = C_{\{\tilde{p}_1, \tilde{p}_2\}} \setminus \partial \tilde{S}.$$

We have:

$$(6.70) \quad \pi(\Psi(J_1)) = \pi(C_{\{\tilde{p}_1, \tilde{p}_2\}}) = C_p^{1)} .$$

(6.71) A point q belongs to C_p if q is generalized pica²⁾ relative to p . For every point $q \in C_p$ there exist exactly two distinct (normalized) minimal joins $c_1[0, d(p, q)]$, $c_2[0, d(p, q)]$ going from p to q and there exists a number $\delta^{3)} \in]0, d(p, q)[$ such that $c_1[0, \delta] = c_2[0, \delta]$ and $c_1[\delta, d(p, q)] \cup c_1[\delta, d(p, q)]$ yields a non-trivial simple loop. The loop $c_1[0, d(p, q)] \cup c_1[0, d(p, q)]$ is homotopic to the (modulo orientation) unique shortest, non-trivial loop with base point p . We have $m = \pi(\tilde{m}) \in C_p$ and $d(p, C_p) = d(p, m)$, m being the mid point of the (modulo orientation) unique shortest non-trivial loop with base point p .

$$(6.72) \quad \pi \circ \Psi : J_1 \rightarrow S$$

is an embedding, the restriction

$\pi \circ \Psi|_{]0, 1[} :]0, 1[\rightarrow S$ is a C^1 -smooth embedding,

$$\pi \circ \Psi(]0, 1[) = C_p \setminus \partial S.$$

1) In the first step for the proof of (6.70) one can show:

(6.70a) There exists $\varepsilon > 0$ such that $\pi(\Psi[0, \varepsilon[)) \subset C_p$.
The proof of (6.70a) uses the same methods as the proof of (6.59c). After this step the proof of (6.70) is literally the same one as the proof of (6.59).

2) Cf. definition 6.5.

3) δ is depending on p and q .

(6.72') Let $q = \pi \circ \Psi(t) \in C_p \setminus \partial S$ then $\frac{d\pi \circ \Psi(t)}{dt}$ bisects the angle built by two initial vectors of those two minimal joins going from q to the point p .

(6.73) If $1 \notin J_1$ then for every sequence t_n in J_1 with $\lim t_n = 1$ $d(\pi \circ \Psi(t_n), p)$ is an unbounded sequence.

(6.74) $\pi \circ \Psi(0) \in T_1$. If $1 \in J_1$ then $\pi \circ \Psi(1) \in \partial S \setminus T_1$.

Combining the preceding main results of case (6.56) and case (6.57) we finish now the proof of theorem 6.5):
The combination of (6.59), (6.61), (6.70), (6.72) proves theorem 6.5a). Further (6.62), (6.63), (6.64), (6.73), (6.74) imply theorem 6.5b), (6.60) and (6.71) imply theorem 6.5c), (6.61') and (6.72') prove theorem 6.5d), (6.56) proves theorem 6.5e).

Hence the proof of theorem 6.5 is complete.

APPENDIX

We give in the appendix technical lemmata which are used in the proofs of major results in this paper. The following lemma A.1 is used in the proof of theorem 3.1. (We realized that H. Federer has given in [29] p. 434 a proof different from ours for another version of lemma A.1.) Lemma A.1 is contained in the subsequent lemma A.1. However, since we use lemma A.1 in our proof of lemma A.1 we start now with the statement and proof of lemma A.1.

Lemma A.1. Let f be a real valued function defined on an open subset O of R^n . Further let $w:O \rightarrow R^n$ be a continuous vector field on O . Now if f is locally Lipschitz continuous and if its gradient at those points where it exists equals the value of the vector field w , then f is a C^1 -smooth function on O and gradient f equals w .

Proof of lemma A.1. Let x_0 be an arbitrary point in O and let us restrict our further considerations to a small compact ball B with center x_0 , such that we have f Lipschitz continuous on B , $B \subset O$. A theorem of Rademacher and Stepanoff see [30] p. 216/218 tells us that f is almost everywhere differentiable in B relative to the n -dimensional Lebesgue measure in R^n . The proof is done if we can show that for any given $\epsilon > 0$ exists a positive number $\delta(\epsilon)$, such that

$$|f(x_0+h) - f(x_0) - w(x_0)h| \cdot \frac{1}{|h|} < \varepsilon$$

if

$$|h| < \delta(\varepsilon).$$

Here h is a vector with an arbitrary direction and $w(x_0)$ is identified with the corresponding linear mapping. First since $w(\cdot)$ is uniform continuous on the compact set B , we can choose $\delta(\varepsilon)$ so small that

$$|w(\bar{x}+\bar{h}) - w(\bar{x})| < \frac{\varepsilon}{4}$$

if

$$|\bar{h}| < \delta(\varepsilon),$$

\bar{x} , $\bar{x}+\bar{h}$ being points in B . Let us take now an arbitrary but for the sequel fixed chosen direction $\frac{h}{|h|}$, $h \neq 0$. We consider a product representation of the n -dimensional Lebesgue measure L^n in R^n relative to the subspace spanned by $\frac{h}{|h|}$ and the hyperspace orthogonal to $\frac{h}{|h|}$. We know that on almost all segments parallel to $\frac{h}{|h|}$ the function f has almost everywhere a gradient in B relative to the 1-dimensional Lebesgue measure on those segments. This is true because for a set of L^n measure zero almost all sections of this set chosen relative to some product representation of L^n must have measure zero relative to the corresponding lower dimensional Lebesgue measure of that representation of L^n ; this is a consequence of a

theorem of Cavalieri-Fubini see e.g. [56] p. 384 or [30] p. 115. Therefore, on a dense subset of segments parallel to $\frac{h}{|h|}$ the function f has L^1 almost everywhere a gradient in B . Now if e.g. on a segment $\tilde{x}_0 + sh$, $0 \leq s \leq 1$, the function f has a gradient for almost all s in $[0,1]$, then the function $s \mapsto \varphi(s) := f(\tilde{x}_0 + sh)$ is C^1 -smooth on $[0,1]$. Namely the function $s \mapsto \varphi(s)$ being Lipschitz continuous can by a well known theorem be represented as an integral of its derivative $\dot{\varphi}(s)$ which coincides here for almost all s in $[0,1]$ with the continuous function $s \mapsto w(\tilde{x}_0 + sh)h$. This holds because we have here

$$\dot{\varphi}(s) = \frac{d}{ds} \Big|_{s=s} f(\tilde{x}_0 + sh) = \dot{f}(\tilde{x}_0 + sh)h = w(\tilde{x}_0 + sh)h,$$

if f has a gradient at $\tilde{x}_0 + sh$. Therefore, we get $\dot{\varphi}(s) = w(\tilde{x}_0 + sh)h$ for all s in $[0,1]$. This yields

$$|f(\tilde{x}_0 + h) - f(\tilde{x}_0) - w(\tilde{x}_0)h| = |\varphi(1) - \varphi(0) - \dot{\varphi}(0)| \leq$$

$$(A.1) \leq \max_{0 \leq s \leq 1} |\dot{\varphi}(s) - \dot{\varphi}(0)| \leq \max_{0 \leq s \leq 1} |w(\tilde{x}_0 + sh) - w(\tilde{x}_0)| |h| \leq \frac{\varepsilon}{4} |h|$$

if $|h| < \delta(\varepsilon)$ and if $|x_0 - \tilde{x}_0|$, $\delta(\varepsilon)$ are both say smaller than half the radius of B . Finally in order to prepare the last step for the desired estimation,

remember once given h , $|h| < \delta(\varepsilon)$ we are free to choose \tilde{x}_0 arbitrary close to x_0 such that the inequality proved above in (A.1) and the following inequalities hold simultaneously. Those are

$$\begin{aligned} |f(x_0+h) - f(\tilde{x}_0+h)| &< \frac{\varepsilon}{4}|h|, \\ |f(x_0) - f(\tilde{x}_0)| &< \frac{\varepsilon}{4}|h| \\ \text{and } |w(x_0) - w(\tilde{x}_0)| &< \frac{\varepsilon}{4}. \end{aligned}$$

Thus we get

$$\begin{aligned} &|f(x_0+h) - f(x_0) - w(x_0)h| \cdot \frac{1}{|h|} = \\ &= |f(\tilde{x}_0+h) - f(\tilde{x}_0) - w(\tilde{x}_0)h + f(x_0+h) - \\ &- f(\tilde{x}_0+h) + f(\tilde{x}_0) - f(x_0) + (w(\tilde{x}_0) - w(x_0))h| \cdot \frac{1}{|h|} \leq \\ &\leq |f(\tilde{x}_0+h) - f(\tilde{x}_0) - w(\tilde{x}_0)h| \cdot \frac{1}{|h|} + |f(x_0+h) - f(\tilde{x}_0+h)| \cdot \frac{1}{|h|} \\ &+ |f(x_0) - f(\tilde{x}_0)| \cdot \frac{1}{|h|} + |w(x_0) - w(\tilde{x}_0)| \leq 4\frac{\varepsilon}{4} = \varepsilon \end{aligned}$$

and we are finished with the proof of lemma A.1.

The subsequent lemma A.1 is used in the proof of theorem 3.1.

Lemma A.1. Let f be a real valued function defined on a locally compact and locally convex body U in \mathbb{R}^n . Further let $w:U \rightarrow \mathbb{R}^n$ be a continuous vector field on U .

Now if f is locally Lipschitz continuous and if its gradient at those points where it exists in $U \setminus \partial U$, equals the value of the vector field w then f is a C^1 -smooth function on U and gradient f equals w . ∂U denotes the boundary of U .

Proof of Lemma A.1: For the concept of the differentiability of functions defined on convex bodies in \mathbb{R}^n we refer to [77] p.17-18. More generally we explain in [77] p. 17-18 that it makes sense to differentiate functions on subsets of \mathbb{R}^n which satisfy a cone condition. By lemma A.1 we know that f is C^1 -smooth on $U \setminus \partial U$. Therefore, it remains to prove that f is differentiable in an arbitrary boundary point $p=0 \in \partial U$ and $(\text{grad } f)(0) = w(0)$. For this, we are done if we prove that there exists a function $[0, R] \ni r \mapsto \psi(r) \in [0, \infty]$ with $\lim_{r \rightarrow 0} \psi(r) = 0$, such that say

$$(A.2) \quad |F(0+h) - F(0) - w(0)h| \leq \psi(|h|) |h|$$

if h is any vector in \mathbb{R}^n with $\{sh/s \in [0, 1]\} \subset U \cap B_R(0)$, $B_R(0)$ a closed ball with center 0 and radius R . Choosing the positive number R sufficiently small we can assure that $B_{3R}(0) \cap U$ is a compact, convex neighbourhood of the point 0 in U . Define

$$(A.3) \quad \psi(r) := \max\{ |w(0) - w(x)| \mid x \in U, |x| \leq 2r \},$$

$|\cdot|$ the Euclidean norm in \mathbb{R}^n . Then $\bar{\psi}$ is monoton increasing,

and $\lim_{r \rightarrow 0} \Psi(r) = 0$.

Let $h \in U \cap B_R(0)$.

Case 1: $h \in \partial U$. Then by convexity $= \{sh \mid s \in [0,1]\} \subset U \setminus \partial U$.

By Lemma A.1' the function F is C^1 -smooth on $U \setminus \partial U$.

Hence the mean value theorem yields

$$\begin{aligned} |F(h) - F(0) - w(0)h| &\leq \sup_{t \in [0,1]} |DF(th)h - w(0)h| \\ &= \sup_{t \in [0,1]} |w(th)h - w(0)h| \\ (A.4) \qquad \qquad \qquad &\leq \tilde{\Psi}\left(\frac{|h|}{2}\right) |h|. \end{aligned}$$

Case 2: $h \in \partial U$. Let L be a Lipschitz constant for the restriction of F on $U \cap B_R(0)$. Choose $\tilde{h} \in (U \cap B_R(0)) \setminus \partial U$ such that

$$|\tilde{h} - h| \leq \min \left\{ |h|, \frac{|h|^2}{L + w(0) + 1} \right\}.$$

Then

$$\begin{aligned} F(h) - F(0) - w(0)h &\leq |F(h) - F(\tilde{h})| \\ &\quad + |F(\tilde{h}) - F(0) - w(0)\tilde{h}| \\ &\quad + |w(0)\tilde{h} - w(0)h| \\ (A.4) \qquad \qquad \qquad &\leq L|h - \tilde{h}| + \Psi\left(\frac{|\tilde{h}|}{2}\right)|\tilde{h}| + |w(0)| |h - \tilde{h}| \\ &\leq |h|^2 + \bar{\Psi}(|h|) \cdot 2|h| \end{aligned}$$

$$(A.5) \qquad \qquad \qquad \leq (|h| + 2\bar{\Psi}(|h|)) |h|.$$

Hence defining $\Psi(|h|) := (|h| + 2\bar{\Psi}(|h|))$

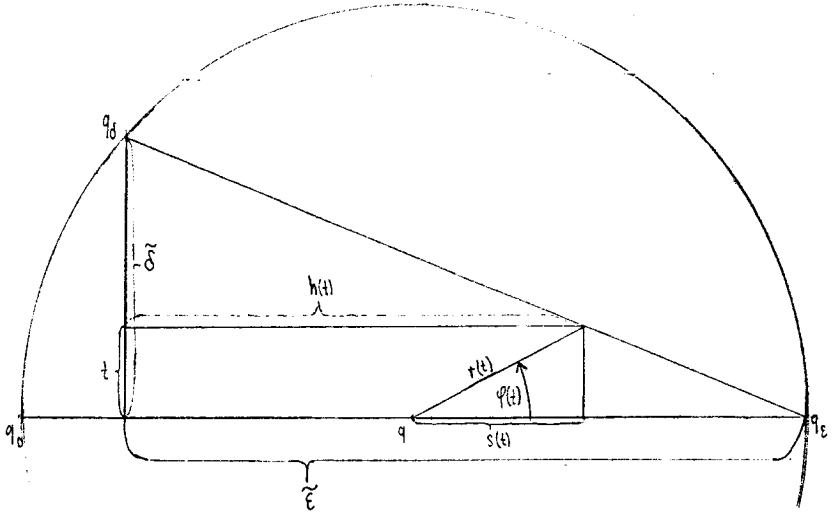
and using (A.4), (A.5) and the monotony of $\bar{\Psi}$ it is now obvious that the proof of Lemma A.1 is complete.

Lemma A.2. Let C be any set in a unbordered complete Riemannian manifold. Assume there exists $\bar{\varepsilon} > 0$ such that for all $q \in C$ the exponential map \exp_q is a diffeomorphism on the Euclidean ball $B_{\bar{\varepsilon}}(0)$. Then we have a continuous function $C \times [0, \bar{\varepsilon}] \ni (q, \varepsilon) \mapsto K(q, \varepsilon) \in [0, \infty[$ with the property described below. Namely let $B_{\varepsilon}(q)$ be a representation in Riemannian normal coordinates for a distance ball with radius ε and center $q \in C$, $|\cdot|$ the Euclidean norm related to this chart. Further let q_0, q_ε be points in $B_{\varepsilon}(q)$ with $|q_0| = |q_\varepsilon| = \varepsilon$ such that the minimal geodesic from q_0 to q_ε meets $q=0$. Then for any point $q_\delta \in B_{\varepsilon}(q)$ with $|q_\delta| = \varepsilon$, $\delta := |q_0 - q_\delta|$ we get

$$d(q_\delta, q_\varepsilon) \leq |q_\delta - q_\varepsilon| + 2K(q, \varepsilon) \varepsilon \delta^2.$$

Therefore defining $\bar{K}(q) := \max \{2K(q, \varepsilon) / \varepsilon \in [0, \bar{\varepsilon}]\}$ we have $d(q_\delta, q_\varepsilon) \leq |q_\delta - q_\varepsilon| + \bar{K}(q) \varepsilon \delta^2$.

Proof. We estimate the Riemannian length of the Euclidean segment connecting q_ε and q_δ . We shall use a polar-coordinate representation of this segment and introduce notations explained in the following figure.



Let $t \rightarrow (\varphi(t), r(t))$. $0 \leq t \leq \delta$ be a polar coordinate representation of the Euclidean segment $c(t)$ in $B(q)$ joining q_ϵ with q_δ the polar coordinate representation chosen relative to the Euclidean 2-plane P_δ containing the points $q_\delta, q=0, q_\epsilon$. Denote the Riemannian length of the segment $c[0, \delta]$ by $L_R(\delta, \epsilon)$ or L_R in short. Then we have

$$(A.6) \quad d(q_\delta, q_\epsilon) \leq L_R = \int_0^\delta \sqrt{\dot{r}^2(t) + g_{22}(r(t), \varphi(t)) \dot{\varphi}^2(t)} dt$$

with $g_{22}(\)$

related to the metric in the surface $\exp_q(P_\delta)$. We assume now the validity of the following inequality

$$(*) \quad g_{22}(r(t), \varphi(t)) \leq r^2(t) + K(q, \epsilon) r^4(t)$$

with $0 \leq K(q, \varepsilon)$ independent of the 2-plane P_δ containing q . Using inequality (*) which we shall show at the end of this proof, we get from (A.6) the following inequalities

$$(A.7) \quad L_R \leq \int_0^{\delta} \sqrt{\dot{r}^2 + r^2 \dot{\phi}^2 + r^4 K(q, \varepsilon) \phi^2} dt \leq \int_0^{\delta} \sqrt{\dot{r}^2 + r^2 \dot{\phi}^2} + \frac{r^4 \frac{1}{2} K \phi^2}{\sqrt{\dot{r}^2 + r^2 \dot{\phi}^2}} dt$$

Note in order to abbreviate the notation we write K instead of $K(q, \varepsilon)$ and we omit the argument variable t .

The last inequality is immediate from the mean value theorem since $\frac{d\sqrt{x}}{dx}$ is decreasing for $x \geq 0$.

Namely if we apply in (A.7) the meanvalue theorem to the first integrand $\sqrt{a+h}$, $a := \dot{r}^2 + r^2 \dot{\phi}^2$,

$$h := r^4 \frac{1}{2} K \phi^2 \quad \text{we get} \quad \sqrt{a+h} = \sqrt{a} + \frac{d\sqrt{x}}{dx}(a+\alpha h) \quad h \leq \sqrt{a} + \frac{d\sqrt{x}}{dx}(a)h,$$

α a certain number in $[0, 1]$. This proves (A.7). The second inequality in (A.7) leads to

$$L_R \leq |q_\delta - q_\varepsilon| + \int_0^{\delta} \frac{1}{\sqrt{1+\dot{h}^2}(t)} (r^4 \frac{1}{2} K \phi^2) dt$$

because we have in appropriately choosen Euclidean coordinates (see figure) $(t, h(t)) = C(t)$ and therefore

$$\sqrt{1+\dot{h}^2}(t) = \left| \frac{d}{dt} C(t) \right| = \sqrt{\dot{r}^2 + r^2 \dot{\phi}^2}. \quad \text{Now a look at}$$

the figure tells that $\dot{h}(t) = -\frac{\varepsilon}{\delta}$

and $|q_\delta - q_\epsilon| = \sqrt{\frac{\tilde{\nu}}{\delta^2 + \tilde{\epsilon}^2}}$. This gives

$$\frac{\gamma}{|q_\delta - q_\epsilon|} = \frac{\gamma}{\sqrt{\frac{\tilde{\nu}}{\delta^2 + \tilde{\epsilon}^2}}} = \frac{1}{\sqrt{1 + (\tilde{\epsilon}/\delta)^2}} = \frac{1}{\sqrt{1 + (\tilde{\kappa}(t))^2}}$$

This yields together with the last inequality the following

$$(A.8) \quad L_R \leq |q_\delta - q_\epsilon| + \frac{\tilde{\nu}}{\delta} \frac{1}{2} K \int_0^\gamma r^2 (r^2 \dot{\phi}^2) dt.$$

Using again $\sqrt{\dot{r}^2 + r^2 \dot{\phi}^2} = \sqrt{1 + \tilde{\kappa}^2} = \sqrt{1 + (\tilde{\epsilon}/\delta)^2}$ we get $r^2 \dot{\phi}^2 = (1 + (\tilde{\epsilon}/\delta)^2) - \dot{r}^2$. Thus we have

$$(A.9) \quad \int_0^\gamma r^2 (r^2 \dot{\phi}^2) dt = \int_0^\gamma r^2 (1 + (\tilde{\epsilon}/\delta)^2 - \dot{r}^2) dt.$$

We bring the integrand on the right hand side of (A.9) now into a form which is easier to compute. For this

$$r^2(t) = s^2(t) + t^2 \text{ leads to } \dot{r}(t) = \frac{s(t)\dot{s}(t) + t}{\sqrt{s^2(t) + t^2}}.$$

Further since obviously $s(t) = \epsilon - \frac{\tilde{\epsilon}}{\tilde{\nu}} t$ we have $\dot{s}(t) = -\frac{\tilde{\epsilon}}{\tilde{\nu}}$. Now we get combining these facts and omitting the argument variable t in the subsequent expressions

$$(A.10) \quad \begin{aligned} \dot{r}^2 - \left(\frac{\tilde{\epsilon}}{\tilde{\nu}}\right)^2 &= \dot{r}^2 - \dot{s}^2 = \frac{s^2 \dot{s}^2 + 2ts\dot{s} + t^2 - \dot{s}^2 (s^2 + t^2)}{s^2 + t^2} = \\ &= \frac{\dot{s}t(2s - \dot{s}t) + t^2}{r^2} = \end{aligned}$$

$$\begin{aligned}
 & \left(-\frac{\tilde{\varepsilon}}{\tilde{\delta}}\right)t\left(\varepsilon - t\frac{\tilde{\varepsilon}}{\tilde{\delta}}\right)2 + t\frac{\tilde{\varepsilon}}{\tilde{\delta}} + t^2 \\
 &= \frac{\quad}{r^2} \\
 &= \frac{\left(-\varepsilon/\tilde{\delta}\right)t\left(2\varepsilon - \frac{\tilde{\varepsilon}}{\tilde{\delta}}t\right) + t^2}{r^2} .
 \end{aligned}$$

Therefore combining (A.9) and (A.10) we get

$$\begin{aligned}
 \int_0^{\tilde{\delta}} r^2 (r^2 \dot{\varphi}^2) dt &= \int_0^{\tilde{\delta}} r^2 + (2\varepsilon\tilde{\varepsilon}/\tilde{\delta})t - (\tilde{\varepsilon}/\tilde{\delta})^2 t^2 - t^2 dt \leq \\
 &\leq \varepsilon^2 \tilde{\delta} + \varepsilon \tilde{\varepsilon} \tilde{\delta} .
 \end{aligned}$$

This yields together with (A.8)

$$\begin{aligned}
 L_R &\leq |q_{\tilde{\delta}} - q_{\varepsilon}| + \frac{\tilde{\delta} \frac{1}{2} K \tilde{\delta} (\varepsilon + \varepsilon) \tilde{\varepsilon}}{|q_{\tilde{\delta}} - q_{\varepsilon}|} \leq |q_{\tilde{\delta}} - q_{\varepsilon}| + \tilde{\delta}^2 \frac{1}{2} K (2\varepsilon + \varepsilon) \leq \\
 &\leq |q_{\tilde{\delta}} - q_{\varepsilon}| + \tilde{\delta}^2 K 2\varepsilon
 \end{aligned}$$

because obviously $\tilde{\varepsilon} \leq 2\varepsilon$, $\tilde{\varepsilon} \leq |q_{\tilde{\delta}} - q_{\varepsilon}|$ and $\tilde{\delta} \leq \delta = |q_{\tilde{\delta}} - q_{\delta}|$.

Thus we have proved lemma A.2 assuming the validity of inequality (*).

Proof of inequality (*):

Although inequality (*) is more or less well known we give now a proof for inequality (*). We do this for two reasons. First we do this for the sake of completeness. Secondly we do not know of any reference containing a detailed and explicit argument concerning the

continuity of the function $(q, \varepsilon) \mapsto K(q, \varepsilon)$ in inequality (*). Let $S(t)$, $W(t)$ be a pair of orthogonal vectors in P_δ with $c(t) = (r(t), \varphi(t)) = r(t) S(t)$. Omitting the argument variable t we have

$$(A.11) \quad g_{22}(r, \varphi) = \left| (d\exp_q(rS))rW \right|_{\exp_q(rS)}^2$$

Since $\exp(\cdot)$ is a C^∞ -smooth mapping the map $\psi : TM \oplus TM \rightarrow \mathbb{R}$ defined by

$$\psi(q, x, y) := \left| (d\exp_q(x))y \right|_{\exp_q(x)}^2$$

$x, y \in T_q M$ is a C^∞ -smooth map with three independent (vectorial) variables, and we have

$$(A.12) \quad \psi(q, rS, rW) = \left| (d\exp_q(rS))rW \right|_{\exp_q(rS)}^2 .$$

Now using a Taylor development we get

$$(A.13) \quad \begin{aligned} \psi(q, rS, rW) &= \psi(q, 0, 0) + \psi'(q, 0, 0)(0, rS, rW) + \dots \\ &+ \frac{1}{4!} \psi^{(4)}(q, 0, 0)(0, rS, rW)^4 + \\ &+ \bar{R}(q, rS, rW)(2r)^4 \end{aligned}$$

Here we define $|(0, S, W)| := |S| + |W|$ and $\psi^{(n)}(q, 0, 0)$ is a multilinear map denoting the n -th derivative of

$\psi(\cdot)$ at the point $(q, 0, 0)$ and

$$(0, rS, rW)^n = \underbrace{((0, rS, rW), \dots, (0, rS, rW))}_{n\text{-times}} .$$

If $\psi(\cdot)$ is C^4 -smooth the remainder term $\bar{R}(\cdot)$ is continuous and we have $R(q, 0, 0) = 0$ for all $q \in M$. Now for a fixed given point q and a fixed given orthogonal pair of vectors S, W in $T_q M$ we define

$\bar{\psi}(r) := \psi(q, rS, rW)$ and get

$$(A.14) \quad \left. \frac{d^n \bar{\psi}(r)}{dr^n} \right|_{r=0} = \psi^{(n)}(q, 0, 0) (0, S, W)^n .$$

Using [25] p. 16 we have

$$\psi(r) := 0 + 0 + r^2 + 0 - \frac{1}{3} \langle R_q(W, S)S, W \rangle_q r^4 + O(r^5) ,$$

$R_q(\cdot)$ the Riemannian curvature tensor at the point q .

This gives in combination with (A.11), (A.12), (A.13)

and (A.14)

$$(A.15) \quad g_{22}(r, \varphi) = \psi(q, rS, rW) = r^2 - \frac{1}{3} \langle R_q(W, S)S, W \rangle_q r^4 + \dots \\ + \bar{R}(q, rS, rW) 16r^4 .$$

Defining $K(q, \epsilon) := \max \{ -\frac{1}{3} \langle R_q(W, S)S, W \rangle_q +$

$$+ \bar{R}(q, rS, rW) / 0 \leq r \leq \epsilon ,$$

$$S, W \in T_q M, |S| = |W| = 1, \langle S, W \rangle_q = 0 \}$$

the continuity of the curvature tensor and the continuity of the remainder term $\bar{R}(\)$ assure that this definition of $K(q, \varepsilon)$ makes sense and guarantee the continuity of the function $(q, \varepsilon) \rightarrow K(q, \varepsilon)$. This proves (*) because of (A.15) and finishes the proof of lemma A.2.

Lemma A.3:

Let M be a Riemannian manifold, $q_0 \in M$, and $r, s > 0$ such that the compact geodesic ball $B_{r+s}(q_0) \subset M \setminus \partial M$ is contained in the domain of Riemannian normal coordinates centered at q_0 . Let $|\cdot|$ denote the norm corresponding to these coordinates, and let $\|\cdot\|$ denote the Riemannian norm on the tangent bundle. Then there exists $L > 0$ such that for all $q, \bar{q} \in B_r(q_0)$ and $v \in T_q M$, $\bar{v} \in T_{\bar{q}} M$ with $\|v\|, \|\bar{v}\| < s$ we have

$$d(\exp_q(v), \exp_{\bar{q}}(\bar{v})) \leq L(d(q, \bar{q}) + |v - \bar{v}|).$$

Proof:

Let $f, F > 0$ be such that

$$f |x-y| \leq d(x,y) \leq F|x-y|$$

for all $x, y \in B_{r+s}(q_0)$, see proposition 4.1. Then using a standard estimate from the theory of ordinary differential equations we obtain for some constant $C > 0$

$$\begin{aligned} d(\exp_q(v), \exp_{\bar{q}}(\bar{v})) &\leq F |\exp_q(v) - \exp_{\bar{q}}(\bar{v})| \\ &\leq FC(|q - \bar{q}| + |v - \bar{v}|) \\ &\leq FC \left(\frac{1}{f} d(q, \bar{q}) + |v - \bar{v}| \right) \\ &\leq FC \left(1 + \frac{1}{f} \right) (d(q, \bar{q}) + |v - \bar{v}|) \end{aligned}$$

Lemma A.4: Let B be a compact subset in a bordered or unbordered Riemannian manifold M . Assume there exists $\epsilon > 0$ such that $\exp_q : K_\epsilon(0) \xrightarrow{1)} \exp_q(K_\epsilon(0)) \subset M \setminus \partial M$ is a diffeomorphism for all $q \in B$. Then there exist numbers $G, F > 0$ such that the following holds: Let q be any point in B then for all $x, y \in T_q M$ with $|x|, |y| \leq \epsilon$ we have

$$(*) \quad G d(\exp_q(x), \exp_q(y)) \leq |x-y| \leq F d(\exp_q(x), \exp_q(y)),$$

|.| the norm in $T_q M$.

Proof of Lemma A.4: Clearly

$$d(\exp_q(x), \exp_q(y)) \leq \text{length} \{ \exp_q(x+t(y-x)) \mid t \in [0,1] \} =: L.$$

Let $c(t) := x + t(y-x)$ then

$$L = \int_0^1 \left| \left(\text{Dexp}_q(c(t)) \right) \dot{c}(t) \right|_{\exp_q(c(t))} dt$$

$$\leq \left(\int_0^1 \left| \text{Dexp}_q(c(t)) \right|_{\exp_q(c(t))} |\dot{c}(t)| dt \right) =: L,$$

$\left| \text{Dexp}_q(c(t)) \right|_{\exp_q(c(t))}$ the norm of the linear map

$\text{Dexp}_q(c(t)) : T_{c(t)} TM \longrightarrow T_{\exp_q(c(t))} M$. Since B is compact

there exists

1) $K_\epsilon(0) = \{v \in T_q M \mid |v|_q \leq \epsilon\}$

2) Here $\dot{c}(t) = \frac{dc(t)}{dt}$ and $\text{Dexp}_q(c(t))$ denotes the differential of the exponential map at the point $c(t) \in T_q M$.

$$\alpha := \max \{ |D\exp_q(v)|_{\exp_q(v)} / |v_q| \mid v_q \in TM, |v_q| \leq \varepsilon \}.$$

Thus

$$L \leq \alpha \int_0^1 |\dot{c}(t)| dt = \alpha |x - y|.$$

Therefore

$$\frac{1}{\alpha} d(\exp_q(x), \exp_q(y)) \leq |x - y|.$$

Since B is compact and as $\exp_q(K_\varepsilon(0)) \subset M \setminus \partial M$ for all $q \in B$ it is clear that there exists a positive number δ such that

$$\exp_q : K_{\varepsilon+\delta}(0) \longrightarrow \exp_q(K_{\varepsilon+\delta}(0)) \subset M \setminus \partial M$$

is a diffeomorphism for all $q \in B$. Therefore it is easily seen that

$$d(\exp_q(x), \exp_q(y)) \geq \min \left\{ 2\delta, \int_0^1 \ell |\dot{c}(t)| dt \right\},$$

if

$$\ell := \min \{ \min \{ |(D\exp_q(v)) w|_{\exp_q(v)} / |w| \mid |w|=1 \} / |v_q| \mid v_q \in TM, q \in B, |v_q| \leq \varepsilon \}.$$

Thus we get for $x, y \in \{v_q \in TM \mid |v_q| \leq \varepsilon, q \in B\}$

$$\begin{aligned} d(\exp_q(x), \exp_q(y)) &\geq \min \{ 2\delta, \ell |x-y| \} \geq \\ &\geq \min \left\{ 2\delta \frac{|x-y|}{2\varepsilon}, \ell |x-y| \right\} \\ &\geq \min \left\{ \frac{\delta}{\varepsilon}, \ell \right\} |x-y|. \end{aligned}$$

1) The number ℓ is positive because $D\exp_q(v)$ is not singular for all $v_q \in TM$ with $q \in B$ and $|v_q| \leq \varepsilon$.

Therefore defining $G := \frac{1}{\alpha}$, $F := \frac{1}{\min\{\frac{\delta}{\varepsilon}, \ell\}}$

the proof of Lemma A.4 is complete.

Lemma A.5: Let $B_r(q_0) \subset M \setminus \partial M$ and assume that $B_r(q_0)$ is contained in a domain of Riemannian normal coordinates centered at q_0 . Let $\bar{r} \in]0, r[$ and $\delta \in]0, r - \bar{r}[$ be such that for all $q_1 \in B_{\bar{r}}(q_0)$ the ball $B_\delta(q_1)$ is contained in a domain of Riemannian normal coordinates centered at q_1 . Then there exists numbers L, G such that the following holds:

For any $q_1 \in B_{\bar{r}}(q_0)$ and unit tangent vectors v, \bar{v} at points $q, \bar{q} \in B_\delta(q_1)$

$$|v - \bar{v}|_0 \leq \binom{1}{L} |v - \bar{v}|_1 + Gd(q, \bar{q}) .$$

Here $|\cdot|_j$ denotes the "norm" related to the normal coordinates centered at $q_j \in \{q_0, q_1\}$.

1) Being more pedantic one could indicate using different notations that we have here different representations of the tangent vectors, these representations being related to the corresponding coordinates.

Proof of Lemma A.5: Let

$$\phi(.,.) := \exp_{q_0}^{-1} \circ \exp_{(.)}(.): \{w_q \in TB_{\tilde{r}}(q_0) / |w_q|_q \leq \delta\} \rightarrow K_r(q_0)$$

Here we define for every $q_1 \in B_r(q_0)$

$$\phi(q_1, .) := \exp_{q_0}^{-1} \circ \exp_{q_1}(.): K_\delta(q_1) \rightarrow K_r(q_0),$$

with $K_\delta(q_1) := \{v \in T_{q_1} M / |v|_{q_1} \leq \delta\}$,

$$K_r(q_0) := \{v \in T_{q_0} M / |v|_{q_0} \leq r\}.$$

Clearly $|\cdot|_q, |\cdot|_{q_1}, |\cdot|_{q_0}$ denote here the norms induced by the Riemannian metric on the related tangent spaces.

We identify $K_\delta(q_1)$ with $B_{\tilde{\delta}}(q_1)$ and $K_r(q_0)$ with

$B_r(q_0)$. Assume that $c_q(t), c_{\bar{q}}(t)$ are normalized paths with $c_q(0) = q, c_{\bar{q}}(0) = \bar{q}$. $\{q, \bar{q}\} \subset B_\delta(q_1)$ and abbreviate $v = \frac{d}{dt}c_q(t)|_{t=0}, \bar{v} = \frac{d}{dt}c_{\bar{q}}(t)|_{t=0}$.

Let us now denote with v, \bar{v} the representations of those tangent vectors relative to the normal coordinates with center q_1 . Then

$$\partial_2 \phi(q_1, q)(v) := \left. \frac{d\phi(q_1, c_q(t))}{dt} \right|_{t=0}$$

is the representation of v relative to the normal coordinates with center q_0 and $\partial_2 \phi(q_1, q)^{1)}$ is the derivative of

$\phi(.,.)$ relative to the fiber variable. We denote with

$|\partial_2 \phi(q_1, q)|$ the norm of the linear map $\partial_2 \phi(q_1, q)$ and

with $\partial_2^2 \phi(q_1, q)$ the second derivative of $\phi(.,.)$ relative

to the fiber variable, $|\partial_2^2 \phi(q_1, q)|$ the norm of the

bilinear map $\partial_2^2 \phi(q_1, q)$. Therefore

1) Precisely, put $\Psi := \phi(q_1, .) : K_\delta(q_1) \rightarrow K_r(q_0)$. Then Ψ is a map between two normed vector spaces and $\partial_2 \phi(q_1, q)$ is the derivative of Ψ at the point q .

$$\begin{aligned}
 \left| v - \bar{v} \right|_0 &= \left| \partial_2 \phi(q_1, q) (v) - \partial_2 \phi(q_1, \bar{q}) (\bar{v}) \right|_0 = \\
 &= \left| \partial_2 \phi(q_1, q) (v - \bar{v}) + (\partial_2 \phi(q_1, q) - \partial_2 \phi(q_1, \bar{q})) (v) \right|_0 \leq \\
 &\leq \left| \partial_2 \phi(q_1, q) \right| \left| v - \bar{v} \right|_1 + \left| \partial_2 \phi(q_1, q) - \partial_2 \phi(q_1, \bar{q}) \right| \left| v \right|_1 \leq \\
 &\leq \left| \partial_2 \phi(q_1, q) \right| \left| v - \bar{v} \right|_1 + \max_{0 \leq t \leq 1} \left| \partial_2^2 \phi(q_1, q + t(\bar{q} - q)) \right| \left| q - \bar{q} \right|_1 \left| v \right|_1 .
 \end{aligned}$$

Using that $\phi(\cdot)$ is a C^2 -smooth mapping we can define

$$L_0 := \max \{ \left| \partial_2 \phi(q_1, q) \right| / q \in K_\delta(q_1), q_1 \in B_{\bar{r}}(q_0) \}$$

and

$$\tilde{G}_0 := \max \{ \left| \partial_2^2 \phi(q_1, q) \right| / q \in K_\delta(q_1), q_1 \in B_{\bar{r}}(q_0) \} .$$

By Lemma A.4 there exists a number F_0 such that for every

$$q_1 \in B_{\bar{r}}(q_0) \quad \left| q - \bar{q} \right|_1 \leq F_0 d(q, \bar{q})$$

holds for all $q, \bar{q} \in B_\delta(q_1)$. It remains to estimate $\left| v \right|_1$.

Recall $c_q(t)$ is a normalized path. Thus $\left| v \right|_q = 1$.

Hence there exists a number \tilde{f}_0 such that $\tilde{f}_0 \geq \left| v \right|_0$.

Thus

$$\left| v \right|_0 = \left| \partial_2 \phi(q_1, q) (v) \right|_0 \leq \tilde{f}_0 .$$

Now

$$\phi(q_1, \cdot) : K_\delta(q_1) \rightarrow \phi(K_\delta(q_1))$$

is a diffeomorphism for all $q_1 \in B_{\bar{r}}(q_0)$. Therefore there exists

$$H_0 := \min \{ \left| \partial_2 \phi(q_1, q) (v) \right|_0 / \left| v \right|_1 = 1, v \in E^n,$$

$$q \in K_\delta(q_1), q_1 \in B_{\bar{r}}(q_0) \} ,$$

$(E^n, | \cdot |_1)$ the normed vector space containing $K_\delta(q_1)$.

Thus

$$|\partial_2 \phi(q_1, q) (v)|_0 \geq H_0 |v|_1 .$$

Hence

$$|v|_1 \leq \frac{f_0}{H} .$$

We abbreviate

$$G := G_0 F_0 \frac{f_0}{H} .$$

Using the just defined constants we get

$$\begin{aligned} |\partial_2 \phi(q_1, q) : |v - \bar{v}|_1 + \max_{0 \leq t \leq 1} |\partial_2^2 \phi(q_1, q+t(\bar{q}-q))| |q-\bar{q}|_1 |v|_1 &\leq \\ &\leq L_0 |v - \bar{v}|_1 + G d(q, \bar{q}) . \end{aligned}$$

This proves Lemma A.5.

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