Surface Curve Design by Orthogonal Projection of Space Curves Onto Free-Form Surfaces

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A novel technique for designing curves on surfaces is presented. The design specifications for this technique derive from other works on curvature continuous surface fairing. Briefly stated, the technique must provide a computationally efficient method for the design of surface curves that is applicable to a very general class of surface formulations. It must also provide means to define a smooth natural map relating two or more surface curves.

The resulting technique is formulated as a geometric construction that maps a space curve onto a surface curve. It is designed to be coordinate independent and provides isoparametric maps for multiple surface curves. Generality of the formulation is attained by solving a tensorial differential equation formulated in terms of local differential properties of the surfaces. For an implicit surface, the differential equation is solved in three-space. For a parametric surface the tensorial differential equation is solved in the parametric space associated with the surface representation.

This technique has been tested on a broad class of examples including polynomials, splines, transcendental parametric and implicit surface representations.

1 Introduction

In many applications related to the design, cutting, patching and welding of free-form shell structures, such as in naval and aeronautical architecture or car body design, surfaces have to be cut along pre-defined trimming lines before assembly. Such a trimming line is usually defined in parametric space for parametric surfaces and in three space for implicit surfaces.

This paper introduces a novel method for designing curves on surfaces by mapping a space curve onto a surface curve. This method relies on a novel geometric construction introduced in this paper under the name of *orthogonal projection of a curve onto a surface*. This geometric construction presents many advantages for the purpose of matching adjacent trimmed patches and mapping non-adjacent trimmed patch boundaries for fairing. It offers a uniform design tool that works independently of the surface formulation because it results from a geometric construction rather than a parametric definition. It allows the engineer to design interactively one or more trimming curves on multiple surfaces directly in three-space. Most importantly however, the geometric construction provides a natural smooth map between the space curve and the surface curve or between multiple surface curves. In tested examples, this map has been used to design trimming curves on different surfaces in a natural, i.e. geometric, manner. This map was used in turn to generate blend surface patches that connect with initial surfaces along the trimming curve.

The main objective of this paper is to introduce the orthogonal projection onto curved surfaces as it relates to design of blended trimmed patches. Our focus is on supplying computational methods for the orthogonal projection rather than discussing its mathematical properties.

This paper is organised in three main parts. Section 2 reviews related works that pertain either to the design of trimming curves on surfaces or to the geometric construction introduced here. Section 3 is devoted to introducing the orthogonal projection as an extension of the distance projection, which is already an important facility for analysis and interrogation in computer-aided design. Sections 4 and 5 introduce the computational formulation of the orthogonal projection onto curved surfaces respectively for parametric and implicit surfaces. Section 6 develops further the natural smooth map provided by the orthogonal projection and illustrates how this construction was applied to trimming and blending.

2 Previous Works

Previous works related to the material exposed here come from two different sources. One is related to the design and representation of curves on surfaces. The other is related to previous works in geometric modeling that pertains to the orthogonal projection introduced here.

In the majority of previous works that we surveyed, trimmed patches occur in the context of surface-surface intersection, see for example the work of Casale [2], in which trimming curves are approximated by broken lines in the parametric domain or Patrikalakis et al. [3], in which the trimming line is interpolated by B-splines.

Explicit design of curves on surfaces has been addressed spo-

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radically in the computer aided design community. Bézier [4] designed curves in the parametric domain of a patch. Such a design of a parametric curve in the parametric domain yields a high degree space curve, which was then approximated by a lower degree curve (See Wolter et al. [19].) Hansmann [5] also designed a trimming curve as a B-spline in the parametric domain for the purpose of blending. Barnhill and Ou [6] are pursuing the design of curves and surfaces on surfaces for the purpose of visualization. Commercial CAD packages such as Unigraphics, PATRAN and General Motor's Corporate Graphics System also provide means for designing surface curves but their implementation is not documented.

Distance projection and orthogonal projection of a point onto a surface are presented in detail in Section 3. Both projections map a space point into one or more surface points. However, the former is characterized by a minimum distance criterion, while the latter requires that the projection line be normal to the surface.

To our knowledge, the notion of orthogonal projection in computer aided design appears only in [7], [8] and [9] in the context of variable radius blending. Even though the orthogonal projection onto a curved surface was effectively used in these works, it was not recognized as a geometric construction in its own right and was developed exclusively on the basis of the distance projection. Some developments of the orthogonal projection as well as application to trimming and blending are covered in [10] and [11].

The distance projection problem consists in finding the closest points on a surface to a given point in space. Such a problem occurs frequently in computer aided design and manufacturing. Chen and Ravani [12] for example used the distance projection to map an NC tool path on an offset to the initial surface. Rossignac [13] studied some properties of the distance projection in the context of constructive solid geometry. Finally, Wolter [14] investigated the distance projection in the general context of Riemanian geometry.

3 Distance Projection Versus Orthogonal Projection

Orthogonal projection is related to the known notions of *distance projection of a point onto a set* and *distance between a point* $\mathbf{p}$ *and a set* $\mathbf{S}$ that are investigated extensively by Wolter [14] in the general context of Riemanian manifolds and Rossignac [13] in the context of Constructive Solid Geometry. They are recalled here for the purpose of clarity.

3.1 Distance projection of a point onto a set. By definition, the distance *$d(\mathbf{p},S)$ from a point* $\mathbf{p}$ *to a non-empty set* S is defined

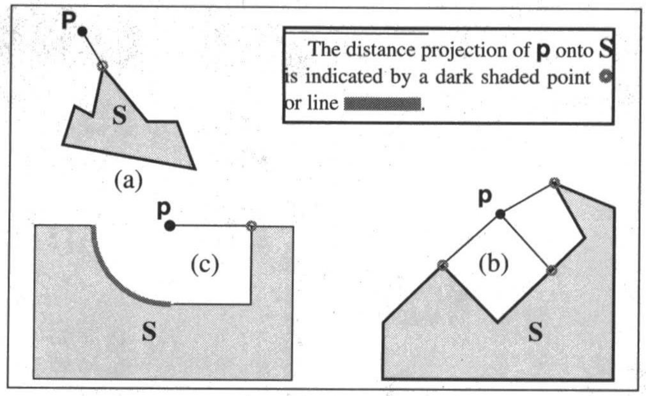

Fig. 1 Two-dimensional examples of projection of a point onto a set: (a) The projection contains only one point, (b) The projection contains more than one discrete point, (c) The projection may contain an infinite number of points.

as the infimum of the distance $d(\mathbf{p},\mathbf{q})$ between $\mathbf{p}$ and any point $\mathbf{q}$ of S.

$$\forall \mathbf{p} \in \Re^3, d(\mathbf{p}, S) = \inf_{\mathbf{q} \in S} d(\mathbf{p}, S) \tag{1}$$

The *distance projection of a point* $\mathbf{p}$ *onto a non-empty set* $\mathbf{S}$ is then defined [14] as the set $\mathbf{P}(\mathbf{p},\mathbf{S})$ of points $\mathbf{q}$ of $cl(\mathbf{S})$ such that $d(\mathbf{p},\mathbf{q})=d(\mathbf{p},\mathbf{S})$.

$$P(\mathbf{p}, \mathbf{S}) = \{\mathbf{q} \in \Re^3 : \mathbf{q} \in cl(\mathbf{S}) \text{and } d(\mathbf{p}, \mathbf{q})\} = d(\mathbf{p}, \mathbf{S}) \tag{2}$$

As illustrated by Figure 1, $\mathbf{P}(\mathbf{p},\mathbf{S})$ may contain a single point, a finite or infinite number of discrete points or an infinite set of points.

3.2 Orthogonal projection of a point onto a surface. We shall now define the *orthogonal projection of a point* $\mathbf{p}$ *onto a surface* $\mathbf{S}$ defined parametrically by a map $\mathbf{q}(u^1,u^2)$ with values in three-space. We assume that the Jacobian of the parametrization is full-rank everywhere. This is equivalent to requiring that the two partial derivatives $\partial\vec{\mathbf{q}}/\partial u^1$ and $\partial\vec{\mathbf{q}}/\partial u^2$ are linearly independent everywhere. *We shall refer to such a surface as being regular.* In this case the cross-product of the partial derivatives is defined everywhere and yields the surface normal. We also require that the surface be second order continuous. This means that the second order partial derivatives of $\mathbf{q}(u^1,u^2)$ are defined and continuous everywhere.

Definition 1: *The orthogonal projection of a point* $\mathbf{p}$ *onto* $\mathbf{S}$ *is the set* $\mathbf{P}^{\perp}(\mathbf{p},\mathbf{S})$ *of points* $\mathbf{q}$ *of* $\mathbf{S}$ *such that the vector* $\overrightarrow{\mathbf{pq}}$ *is nor-*

Nomenclature

- $\in$ = set membership.
- $\subset$ = set inclusion.
- $\setminus$ = set difference.
- $\Re$ = set of real numbers.
- $\mathbf{p}$ = point.
- $\vec{\mathbf{n}}$ = vector.
- $\overrightarrow{\mathbf{pq}}$ = The vector $\mathbf{q}$-$\mathbf{p}$.
- $\|\vec{\mathbf{n}}\|$ = norm of the vector
- $\vec{\mathbf{u}} \bullet \vec{\mathbf{v}}$ = inner product of two vectors.
- C^1 = First order continuity. (The first order partial derivatives exist and are continuous).
- C^2 = Second order continuity. (The second order partial derivatives exist and are continuous).
- $\vec{\mathbf{g}}_i$ = The natural derivatives on a surface $\mathbf{S}$
- δ^{ki} = Kronecker symbol.
- g_{ij} = Components of the first fundamental tensor on a surface $\mathbf{S}$.
- $b_{\alpha\beta}$ = Components of the second fundamental tensor on a surface $\mathbf{S}$.
- $\overrightarrow{\nabla f}$ = The gradient of f.
- $\partial\mathbf{S}$ = Boundary of $\mathbf{S}$.
- $cl(\mathbf{S})$ = Closure of the set $\mathbf{S}$.
- ◆ = end of proof.
- $\inf_{\mathbf{q} \in S} f(\mathbf{q})$ = Infimum of the function $f(\mathbf{q})$ over a set $\mathbf{S}$.

We shall use Einstein's implicit summation convention. Summation applies to identical indices in the same prefix or suffix and in different subscript or superscript position. When applicable, superscript indices on a scalar corresponds to contravariant components, subscript to covariant components. This convention is described in detail in [1]

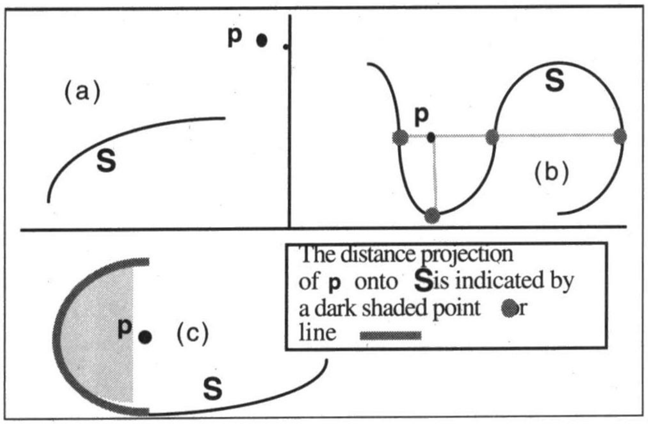

Fig. 2 The orthogonal projection of a point onto a regular surface S may be empty (a), contain many isolated points (b), or an infinite number of points (c), which are not necessarily distance projections of the point.

mal to the surface $\mathbf{S}$ *at point* $\mathbf{q}$.

$$\mathbf{P}^{\perp}(\mathbf{p}, \mathbf{S}) = \left\{ \mathbf{q} \in \mathbf{S} : \overrightarrow{\mathbf{pq}} \bullet \frac{\partial}{\partial u^j} \vec{\mathbf{q}} = 0;\ 1 \le j \le 2 \right\} \qquad (3)$$

Definition 1 distinguishes the notions of distance projection and orthogonal projection. Whereas the distance projection $\mathbf{P}(\mathbf{p},\mathbf{S})$ was shown to be *nonempty* for a closed point set $\mathbf{S}$ in [14], the orthogonal projection $\mathbf{P}^{\perp}(\mathbf{p},\mathbf{S})$ may be empty as illustrated by Figure 2. This can be rephrased as follows: if $\mathbf{S}$ is a closed set then for any given space point $\mathbf{p}$ there always exists a nearest point on $\mathbf{S}$ guaranteeing the distance projection of $\mathbf{p}$ onto $\mathbf{S}$ to be non-empty. However, if $\mathbf{S}$ is a surface patch with boundary it is not guaranteed that there exists any point $\mathbf{q}$ of $\mathbf{S}$ that can be joined to $\mathbf{p}$ along a surface normal at $\mathbf{q}$, see for example Figure 2, case (a). Finally, it is possible for both the orthogonal and the distance projection to contain more than one point. It may contain a major part or even the whole surface, for example take the projection of the center of a sphere onto the sphere itself. These possibilities are illustrated in Figure 2, cases (b) and (c).

Although the orthogonal projection was not addressed as an entity per se in their works, Wolter [14] and Rossignac [13] proved that, if the surface $\mathbf{S}$ admits a first-order continuous local parameterization at a point $\mathbf{q} \in \mathbf{P}(\mathbf{p},\mathbf{S})$, then the distance projection $\mathbf{q}$ of $\mathbf{p}$ onto $\mathbf{S}$ is an orthogonal projection of $\mathbf{p}$ onto $\mathbf{S} \backslash \partial\mathbf{S}$. To be more precise, the distance projection $\mathbf{P}(\mathbf{p},\mathbf{S})$ is a subset of the orthogonal projection $\mathbf{P}^{\perp}(\mathbf{p},\mathbf{S})$. The proof uses the fact that here, the projection $\mathbf{q}$ corresponds to a local extremum of the Euclidean distance $d(\mathbf{p},\mathbf{m})$, $\mathbf{m} \in \mathbf{S}$ at $\mathbf{q}$.

Theorem 1: *If* $\mathbf{S}$ *admits a* C^1 *parametrization and the distance projection* $\mathbf{P}(\mathbf{p},\mathbf{S})$ *is included in the interior of* $\mathbf{S}$, *then* $\mathbf{P}(\mathbf{p},\mathbf{S})$ *is included in* $\mathbf{P}^{\perp}(\mathbf{p},\mathbf{S})$.

$$\mathbf{P}(\mathbf{p}, \mathbf{S}) \subseteq \mathbf{P}^{\perp}(\mathbf{p}, \mathbf{S}) \qquad (4)$$

Proof:

$$\begin{aligned}
\mathbf{q} \in \mathbf{P}(\mathbf{p},\mathbf{S}) &\Rightarrow d(\mathbf{p},\mathbf{q}) \text{ is minimum} \\
&\Rightarrow \overrightarrow{\mathbf{pq}}^{2} \text{ is minimum} \\
&\Rightarrow \overrightarrow{\mathbf{pq}} \bullet \frac{\partial \mathbf{q}}{\partial u_j} = 0 \\
&\Rightarrow \overrightarrow{\mathbf{pq}} \text{ is normal to } \mathbf{S} \text{ at point } \mathbf{q}.
\end{aligned}$$

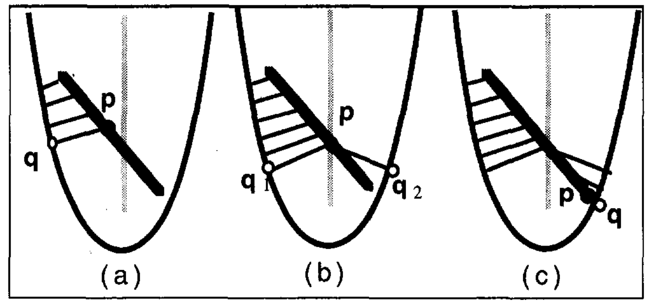

Fig. 3 The distance projection of a point $\mathbf{p}$ onto a parabola contains one point $\mathbf{q}$ when $\mathbf{p}$ is not on the dotted central axis of the parabola. However, when crossing the central axis, the location of the distance projection jumps from one parabola branch to the other. When $\mathbf{p}$ is on the central axis, the distance projection contains two points $\mathbf{q}_1$ and $\mathbf{q}_2$. This example shows that the distance projection $\mathbf{q}$ of a family of points $\mathbf{p}$ cannot be traced continuously on the parabola. Note that the dotted central axis of the parabola is the cut locus of the parabola. The cut locus concept has been investigated by Wolter [14] and [20].

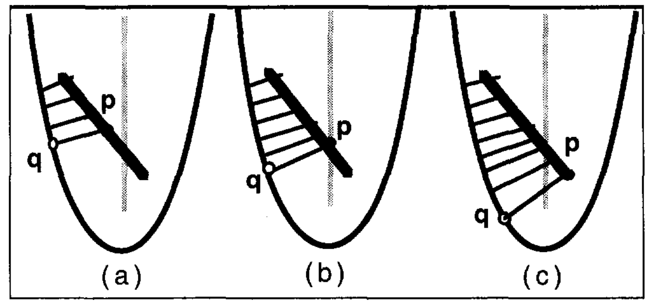

Fig. 4 The orthogonal projection of a point $\mathbf{p}$ onto a parabola is multivalued. It contains points on both branches of the parabola. However, the family of footpoints on each branch can be traced continuously even when the projected point q crosses the dotted central axis of the parabola.

$$\Rightarrow \mathbf{q} \in \mathbf{P}^{\perp}(\mathbf{p},\mathbf{S}) \quad \blacklozenge$$

The converse of Theorem 1 is not true because:

1. Any local extrema of the distance $d(\mathbf{p},\mathbf{m})$, $\mathbf{m} \in \mathbf{S}$, $\mathbf{m} \notin \partial\mathbf{S}$, on a differentiable surface is an orthogonal projection while $d(\mathbf{p},\mathbf{m})$ is not necessarily an infimum.
2. The point $\mathbf{m} \in \mathbf{S}$ on a differentiable surface may be an orthogonal projection while $d(\mathbf{p},\mathbf{m})$ is not even a local extremum (eg. saddle point on the surface).

However, Theorem 1 is a useful property for finding at least one point in the orthogonal projection of $\mathbf{p}$ if the nearest footpoint on $\mathbf{S}$ is not on $\partial\mathbf{S}$.

3.3 Continuity of the orthogonal projection. Wolter [14] as well as Rossignac[13] proved that the distance $d(\mathbf{p},\mathbf{S})$ is a continuous function of $\mathbf{p}$ everywhere. However the distance projection $\mathbf{P}(\mathbf{p},\mathbf{S})$ is not a continuous function of $\mathbf{p}$ everywhere as can be shown with the simple example illustrated by Figure 3. The points in the distance projection are the optimal solutions of a non-convex quadratic optimization problem and are subject to discontinuous changes. Also, the distance projection may change from a single point to an infinite set as illustrated by Figure 1, case (c).

In contrast, the orthogonal projection onto a surface allows to trace footpoints continuously in situation where that continuous tracing is not possible with the distance projection. This is illustrated by the example in Figure 4.

3.4 Orthogonal projective tensor. The orthogonal projective tensor, which is introduced in this section, plays a central role in the rest of this paper. Although it may seem somewhat unrelated to the previous material, its early introduction will simplify the rest of this presentation.

Definition 2: *Let* $\mathbf{p}$ *be a space point,* S *be a* C^2 *continuous regular surface and let* $\mathbf{q}$ *be a point of* S. *The tensor with components* κ_{jk} *defined below is said to be an orthogonal projective tensor of* S *at* $\mathbf{p}$ *if* $\mathbf{q} \in \mathbf{P}^{\perp}(\mathbf{p}, \mathbf{S})$

$$\kappa_{jk} = \frac{\partial \vec{\mathbf{q}}}{\partial u^j} \bullet \frac{\partial \vec{\mathbf{q}}}{\partial u^k} + \overrightarrow{\mathbf{pq}} \bullet \frac{\partial^2 \vec{\mathbf{q}}}{\partial u^j \partial u^k} \tag{5}$$

Theorem 2: *Let* S *be a* C^2 *continuous regular surface. Let* $\rho = \overrightarrow{\mathbf{pq}} \bullet \vec{\mathbf{n}}$ *where* $\vec{\mathbf{n}}$ *is the outward unit normal to the surface* S *at point* $\mathbf{q}$. *Let* g_{ij} *and* b_{ij} *denote respectively the components of the first and second fundamental tensors, then the orthogonal projective tensor is related to the first and second fundamental tensors by the following equation:*

$$\kappa_{ij} = g_{ij} + \rho b_{ij} \tag{6}$$

Proof: Recall that $\vec{g}_i = \frac{\partial \mathbf{q}}{\partial u^i}$, therefore, by definition of the first fundamental tensor we have:

$$g_{ij} = \vec{g}_i \bullet \vec{g}_j = \frac{\partial \mathbf{q}}{\partial u^i} \bullet \frac{\partial \mathbf{q}}{\partial u^j} \tag{7}$$

Since $\overrightarrow{\mathbf{pq}}$ is normal to the surface at $\mathbf{q}$, $\overrightarrow{\mathbf{pq}} = \rho \vec{\mathbf{n}}$, and by definition of the second fundamental tensor, we have:

$$\overrightarrow{\mathbf{pq}} \bullet \frac{\partial^2 \vec{\mathbf{q}}}{\partial u^i \partial u^j} = \rho \vec{\mathbf{n}} \bullet \frac{\partial \vec{g}_i}{\partial u^j} = \rho b_{ij} \quad \blacklozenge \tag{8}$$

Corollary 2.1: *Let* S *be a* C^2 *continuous regular surface. An orthogonal projective tensor of* S *in* $\mathbf{p}$ *is singular —i.e. the matrix* κ_{ij} *is singular— if and only if* $\mathbf{p}$ *is one of the principal centers of curvature of the surface in* $\mathbf{q}$.
Proof: Using equation (6), corollary 2.1 derives from a classical result of tensorial analysis (see for example [1]) ♦

4 Orthogonal Projection of a Space Curve Onto a Parametric Surface

In this section and the next one, we shall develop the notion of orthogonal projection of a space curve onto a surface. As our focus is on supplying computational methods for the orthogonal projection, the discussions of the mathematical conditions for existence, continuity and differentiability of the orthogonal projection curve are not exposed in details here. Suffice to say that these conditions are met provided that the space curve remains close enough to the surface and the projected curve remains in the interior of the surface patch.

Definition 3: *Given a space curve* Γ *and a regular* C^2 *continuous surface* S, *we define the orthogonal projection of* Γ *onto* S *as the set* $\mathbf{P}^{\perp}(\Gamma, \mathbf{S})$ *of points of* S *that are orthogonal projections of at least one point of* Γ.

$$\mathbf{P}^{\perp}(\mathbf{G}, \mathbf{S}) = \{\mathbf{q} \in \mathbf{S} : \exists \mathbf{p} \in \mathbf{G} \text{ such that } \mathbf{q} \in \mathbf{P}^{\perp}(\mathbf{p}, \mathbf{S})\} \tag{9}$$

4.1 Differential equation of the Orthogonal Projection: As we mentioned above in Section 3, the orthogonal projection of a point may be multivalued. For a curve, the orthogonal projection may result in a set of different projection curves on the surface. We are interested in tracing **one** such curve. See for example in Figure 4 the sequence of projection points on a parabolic arc.

In the remainder of this section, it is assumed that Γ is a curve parametrized in t. Consider a point $\mathbf{p}(t)$ of Γ, and an orthogonal projection point $\mathbf{q}(t)$ on the surface S, i.e. $\mathbf{q}(t) \in \mathbf{P}^{\perp}(\mathbf{p}, \mathbf{S})$. The motion of the orthogonal projection $\mathbf{q}(t)$ as $\mathbf{p}(t)$ moves on Γ is a continuous function of $\mathbf{p}(t)$. This continuity statement will be justified a posteriori by solving the differential equation associated with the orthogonal projection.

Let $\{\vec{g}_i\}$ denote the local covariant basis [1] on the surface in curvilinear coordinates $\{u^i\}$ and let the point $\mathbf{q}$ be in the orthogonal projection set of the point $\mathbf{p} \in \Gamma$ onto the surface S. The property of orthogonality translates into a system of 2 equations:

$$\overrightarrow{\mathbf{pq}} \bullet \vec{g}_i = 0. \tag{10}$$

Then taking the derivative of equation (10) with respect to t yields:

$$\frac{d\vec{\mathbf{q}}}{dt} \bullet \vec{g}_i + \overrightarrow{\mathbf{pq}} \bullet \frac{d\vec{g}_i}{dt} = \frac{d\vec{\mathbf{p}}}{dt} \bullet \vec{g}_i \tag{11}$$

The chain rule is used to express $d\vec{\mathbf{q}}/dt$ and $d\vec{g}_i/dt$ in terms of the natural coordinates.

$$\frac{d\vec{\mathbf{q}}}{dt} = \frac{\partial \vec{\mathbf{q}}}{\partial u^j}\frac{du^j}{dt} = \vec{g}_j \frac{du^j}{dt} \tag{12}$$

$$\frac{d\vec{g}_i}{dt} = \frac{\partial \vec{g}_i}{du^j}\frac{du^j}{dt} \tag{13}$$

Substituting (12) and (13) into equation (11), we find that the projection of the first derivatives of $\mathbf{p}(t)$ onto the tangent plane of S at point $\mathbf{q}$ is linearly related to the first derivatives of $\mathbf{q}(t)$.

$$\left(\vec{g}_i \bullet \vec{g}_j + \overrightarrow{\mathbf{pq}} \bullet \frac{\partial \vec{g}_i}{\partial u^j}\right)\frac{du^j}{dt} = \frac{d\vec{\mathbf{p}}}{dt} \bullet \vec{g}_i \tag{14}$$

Recognizing the orthogonal projective tensor from equation (6) in equation (14), the linear relation becomes:

$$\kappa_{ij}\frac{du^j}{dt} = \frac{d\vec{\mathbf{p}}}{dt} \bullet \vec{g}_i \tag{15}$$

Let κ^{ji} be the covariant coordinates of the orthogonal projective tensor [1], i.e. the matrices with components κ_{ij} and κ^{ji} are inverse of each other. Then, if the orthogonal projective tensor is non-singular, the system of equations (15) with unknown du^j/dt can be solved by:

$$\frac{du^j}{dt} = \kappa^{ji}\frac{d\vec{\mathbf{p}}}{dt} \bullet \vec{g}_i \tag{16}$$

Written in expanded form, the system of equations (16) becomes:

$$\frac{du^j}{dt} = \sum_{i=1}^{2} \kappa^{ji}\frac{d\vec{\mathbf{p}}}{dt} \bullet \vec{g}_i,\ j = 1, 2 \tag{17}$$

For the sake of improved readability, a matrix formulation of equation (17) is also provided below. Letting K be the matrix with coefficients κ_{ij}, we have:

$$\begin{bmatrix} \frac{du^1}{dt} \\ \frac{du^2}{dt} \end{bmatrix} = K^{-1} \begin{bmatrix} \frac{d\vec{\mathbf{p}}}{dt} \bullet \vec{g}_1 \\ \frac{d\vec{\mathbf{p}}}{dt} \bullet \vec{g}_2 \end{bmatrix} \tag{18}$$

4.2 Computation of the orthogonal projection. Given equation (16) derived above, the orthogonal projection curve is obtained by solving an initial value problem. One possibility to find an initial point for this problem could use the distance projection of an inital point $\mathbf{p}$ on the space curve Γ. Here one may also be satified with a point $\mathbf{m}_0 \in \mathbf{S}$ being a local minimum for the distance function $d(\mathbf{p},\mathbf{m}_0)$, thereby being an orthogonal projection according to Theorem 1. That local minimum can be obtained by a sampling method combined with a modified Newton-Raphson search.

Once an initial value is selected a marching method (second order Runge-Kutta or Adams-Bashforth with adaptive accuracy control [15]) is used to solve the initial value problem related to the first order system of differential equations (16).

In practically tested situations, the initial value for equation (16) was chosen to be an interior footpoint on the surface patch $\mathbf{S}$. The solution curve is traced until it hits the patch boundary, which is the only part of the projected curve needed for trimming.

It is possible to refine the accuracy with a post-Newton iteration method. The preceding computation yields an array of points. Using for example a cubic B-spline interpolation for those points yields a closed form B-spline approximation of the orthogonal projection curve. The particular approximation method employed was developed by Wolter et al.[19] for the approximation of general univariate functions covering the solution of an initial value problem as a special instance. This approximation can be made extremely accurate, that is better than single precision for most practical purposes. In the tested examples we could always achieve accuracies of 10^{-8} on the interpolation curve if the orthogonal projection were computed with 10^{-11} accuracy, which was always possible. The above mentioned Newton refinement is however rarely necessary because Adams-Bashforth integration yields very accurate results when computations are done in double precision.

5 Orthogonal Projection of a Space Curve Onto an Implicit Surface

In this section we shall develop an equivalent formulation to equation (16) when the surface $\mathbf{S}$ is represented by an implicit equation. Precisely, we consider a C^2-continuous surface S defined in three-space by

$$\mathbf{S} = \{(x^1, x^2, x^3) \in \Re^3 \, ; \, f(x^1, x^2, x^3)=0\} \tag{19}$$

where f is a C^2-continuous real-valued function. We shall assume further the gradient $\overrightarrow{\nabla f}$ does not vanish at any footpoint at which we trace the orthogonal projection. Under this assumption the surface unit normal vector at a point $\mathbf{q} \in \mathbf{S}$ is given by

$$\vec{\mathbf{n}}_{(\mathbf{q})} = \frac{\overrightarrow{\nabla f(\mathbf{q})}}{\left\|\overrightarrow{\nabla f(\mathbf{q})}\right\|}. \tag{20}$$

As in Section 4, we consider the current point $\mathbf{p}(t)$ of a space curve Γ and we trace a footpoint $\mathbf{q}(t)$ with coordinates $(x^1_{(t)}, x^2_{(t)}, x^3_{(t)})$ on the surface $\mathbf{S}$. Letting $\rho(t)$ be the oriented distance between $\mathbf{p}(t)$ and $\mathbf{q}(t)$ measured along the normal vector, we have the following relation defining $\rho(t)$.

$$\overrightarrow{\mathbf{pq}} = \mathbf{q}_{(t)} - \mathbf{p}_{(t)} = -\rho(t)\vec{\mathbf{n}}(\mathbf{q}_{(t)}) \tag{21}$$

A relation between the derivatives of $\mathbf{p}(t)$ and $\mathbf{q}(t)$ is obtained by differentiating the previous equation (21) with respect to the curve parameter t.

$$\mathbf{p}'_{(t)} - \mathbf{q}'_{(t)} = \rho'_{(t)} \; \vec{\mathbf{n}}(\mathbf{q}_{(t)}) + \rho_{(t)} \frac{\partial \vec{\mathbf{n}}}{\partial x^j}\frac{dx^j}{dt} \tag{22}$$

Note that since $\vec{\mathbf{n}}$ is a unit vector normal to the tangent plane, the vector $\partial\vec{\mathbf{n}}/\partial x^j$ is necessarily in the tangent plane because $\partial\vec{\mathbf{n}}/\partial x^j \bullet \vec{\mathbf{n}} = 0$ as can be seen by differentiating $\vec{\mathbf{n}}^2 = 1$.

Rearranging equation (22) and expanding $\partial\vec{\mathbf{n}}/\partial x^j$ yields

$$\mathbf{p}'_{(t)} - \rho'_{(t)} \vec{\mathbf{n}}(\mathbf{q}_{(t)}) = \mathbf{q}'_{(t)} + \rho(t)\left(\frac{\partial\left(\frac{1}{\left\|\overrightarrow{\nabla f(\mathbf{q})}\right\|}\right)}{\partial x^j}\overrightarrow{\nabla f(\mathbf{q})} + \frac{1}{\left\|\overrightarrow{\nabla f(\mathbf{q})}\right\|}\frac{\partial\overrightarrow{\nabla f(\mathbf{q})}}{\partial x^j}\right)\frac{dx^j}{dt} \tag{23}$$

Projecting both sides of equation (23) onto the tangent plane to $\mathbf{S}$ at point $\mathbf{q}(t)$ simplifies the equation because $\vec{\mathbf{n}}(\mathbf{q}(t))$ and $\overrightarrow{\nabla f(\mathbf{q})}$ are vectors normal to the tangent plane.

$$\mathbf{p}'_{(t)} - (\mathbf{p}'_{(t)} \bullet \vec{\mathbf{n}})\vec{\mathbf{n}} = \mathbf{q}'_{(t)} + \frac{\rho(t)}{\left\|\overrightarrow{\nabla f(\mathbf{q})}\right\|}\left(\frac{\partial\overrightarrow{\nabla f(\mathbf{q})}}{\partial x^j} - \left(\frac{\partial\overrightarrow{\nabla f(\mathbf{q})}}{\partial x^j}\bullet\vec{\mathbf{n}}\right)\vec{\mathbf{n}}\right)\frac{dx^j}{dt} \tag{24}$$

Note that we have subtracted the normal components of those vectors that are not contained in the tangent plane.

Recall that the coordinates of $\mathbf{q}'(t)$ are dx^j/dt. Hence, letting $\mathbf{B}$ be the matrix with column vectors

$$\mathbf{B}_j = \frac{1}{\left\|\overrightarrow{\nabla f(\mathbf{q})}\right\|}\left(\left(\frac{\partial\overrightarrow{\nabla f(\mathbf{q})}}{\partial x^j}\bullet\vec{\mathbf{n}}\right)\vec{\mathbf{n}} - \frac{\partial\overrightarrow{\nabla f(\mathbf{q})}}{\partial x^j}\right), \tag{25}$$

and using the identity matrix $\mathbf{I}$, the right hand side of equation (24) can be reformulated as follows:

$$\mathbf{p}'_{(t)} - (\mathbf{p}'_{(t)} \bullet \vec{\mathbf{n}})\vec{\mathbf{n}} = [\mathbf{I} - \rho(t)\mathbf{B}]\mathbf{q}'_{(t)}. \tag{26}$$

Note that the linear map represented by matrix $\mathbf{B}$ is self-adjoint. Letting n^k be the coordinates of the normal vector $\vec{\mathbf{n}}$, the components of $\mathbf{B}$ are:

$$B^i_j = \frac{1}{\left\|\overrightarrow{\nabla f(\mathbf{q})}\right\|}\left(\frac{\partial^2 f(q)}{\partial x^k \partial x^j}n^k n^i - \frac{\partial^2 f(q)}{\partial x^i \partial x^j}\right), \tag{27}$$

which can be re-written in a more concise form by introducing the Kronecker symbol δ^{ki}:

$$B^i_j = \frac{1}{\left\|\overrightarrow{\nabla f(\mathbf{q})}\right\|}\left(\frac{\partial^2 f(q)}{\partial x^k \partial x^j}n^k n^i - \frac{\partial^2 f(q)}{\partial x^k \partial x^j}\delta^{ki}\right), \tag{28}$$

and factoring out the second derivatives:

$$B^i_j = \frac{1}{\left\|\overrightarrow{\nabla f(\mathbf{q})}\right\|}\frac{\partial^2 f(q)}{\partial x^k \partial x^j}(n^k n^i - \delta^{ki}). \tag{29}$$

Also note that the matrix $\mathbf{B}$ has rank 2 because all its column vectors are in the tangent plane. Because of this and the self-adjointness, it has at most two non-trivial real eigenvalues $1/\rho_1$ and $1/\rho_2$. It will not be proved here that these eigenvalues are the principal curvatures of the surface at point $\mathbf{q}$.

The tensor

$$\mathbf{K} = \mathbf{I} - \rho(t)\,\mathbf{B} \tag{30}$$

is the formulation of the orthogonal projective tensor in the implicit case. Using this notation equation (26) becomes:

$$\mathbf{p}'_{(t)} - (\mathbf{p}'_{(t)} \bullet \vec{\mathbf{n}})\vec{\mathbf{n}} = \mathbf{K}\mathbf{q}'_{(t)} \ . \tag{31}$$

Assuming that the matrix $\mathbf{K}$ is non-singular, the projected curve is determined by solving the system of differential equations

$$\mathbf{q}'_{(t)} = \mathbf{K}^{-1}(\mathbf{p}'_{(t)} - (\mathbf{p}'_{(t)} \bullet \vec{\mathbf{n}})\vec{\mathbf{n}}\ ). \tag{32}$$

The non-singularity of the matrix $\mathbf{K}$ is guaranteed provided that $\mathbf{p}$ is not a center of curvature of the surface $\mathbf{S}$ at point $\mathbf{q}$. Hence again as in the parametric case, the differential equation is well defined if the projected curve does not pass through curvature centers of the surface.

5.1 Computation of the orthogonal projection. Computation of the orthogonal projection of a parametric curve onto an implicitly defined surface is very similar to its parametric counterpart. The projection curve is obtained by solving the initial value problem defined by equation (32) and an initial point obtained as in Section 4. Like in the parametric case a highly accurate B-Spline approximation curve of the orthogonal projection curve can be obtained using the general method described by Wolter et al [19]

The main difference between computations in sections 4 and 5 is that the differential equation is solved in 2-space for parametric surfaces and in 3-space for implicit surfaces. The implicit surface case requires resolution of a linear system of three equations in three unknowns at each step versus the resolution of a linear system of two equations in two unknowns in the parametric surface case. At this point, it is worthwhile mentioning that we give equations (16) and (32) because we want to formulate an explicit first order system of ordinary differential equations. In both the implicit and the parametric case however, the solution to equations (16) and (32) may be computed without computing the inverse of the orthogonal projective tensor. Numerical solutions that do not require inversion of the orthogonal projective tensor are especially valuable in the implicit case. Since equation (32) is a system of three equations in three unknowns, it requires more work to invert the matrix $\mathbf{K}$ than to invert the two by two matrix $\mathbf{K}$ in the parametric case.

6 Application to Trimming and Blending

The approach offered by orthogonal projection onto curved surfaces presents many advantages for the design of trimmed patches. Orthogonal projection is a purely geometric construction. Therefore it is both coordinate and representation independent. But the most important contribution of this design tool is that it provides a natural smooth map between a space curve and a surface curve. This map has been used effectively to map multiple surface curves, as in Figure 5 for example. These surface curves are used for trimming original patches while the map provides a natural correspondence between non-adjacent boundaries of trimmed patches. This correspondance between different trimming curves being orthogonal projection curves is obtained because the projection curves inherit their parametrization from the projected space curve which may be the same one used for the projection onto different surface patches (see Figure 5.)

Trimmed patches appear primarily in the context of surface intersections, see for example [2]. The orthogonal projection offers a design tool that allows definition of a trimming curve independently of surfaces intersection but still retains a map between two different surface curves. This property can be used advanta-

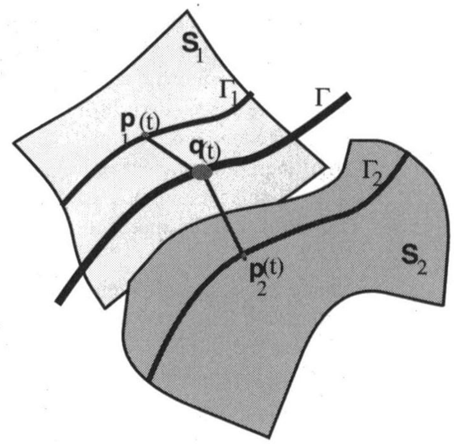

Fig. 5 A space curve Γ is mapped into two surface curves Γ_1 and Γ_2 respectively on surfaces $\mathbf{S}_1$ and $\mathbf{S}_2$. The orthogonal projection maps a point $\mathbf{q}(t)$ of the space curve into its images $\mathbf{p}_1(t)$ and $\mathbf{p}_2(t)$ respectively on $\mathbf{S}_1$ and $\mathbf{S}_2$. This map can be used in turn to map $\mathbf{p}_1(t)$ into $\mathbf{p}_2(t)$.

geously to extend the realm of trimmed patches into surface blending, as we shall now demonstrate.

Assume that we are given a space curve Γ and two surface patches $\mathbf{S}_1$ and $\mathbf{S}_2$ with respective parametrizations $\mathbf{p}_1(u^1,v^1)$ and $\mathbf{p}_2(u^2,v^2)$. A point $\mathbf{q}(t)$ on the curve Γ can be mapped into two points $\mathbf{p}_1(t)=\mathbf{p}_1(u^1(t),v^1(t))$ and $\mathbf{p}_2(t)= \mathbf{p}_2(u^2(t),v^2(t))$ respectively on $\mathbf{S}_1$ and $\mathbf{S}_2$ (Figure 5). Actually, when a surface $\mathbf{S}$ is defined parametrically by $\mathbf{p}(u,v)$ the orthogonal projection maps the point $\mathbf{q}(t)$ directly into the pre-image $(u(t),v(t))$ of the orthogonal projection point $\mathbf{p}(t)$ in parametric space. In both the implicit and the parametric cases, we have the following: If the point $\mathbf{q}(t)$ does not pass through any curvature center of the two surfaces then the surface curves Γ_1 and Γ_2 respectively traced by $\mathbf{p}_1(t)$ and $\mathbf{p}_2(t)$ on $\mathbf{S}_1$ and $\mathbf{S}_2$ are C1-smooth if the patches are C^2-smooth and the parametrization $\mathbf{q}(t)$ of the curve Γ is C^1-smooth. The correspondence between the two simultaneously traced projection curves $\mathbf{p}_1(t)$ and $\mathbf{p}_2(t)$ can be described by a smooth map $\mathbf{m}$ mapping the surface curve Γ_1 into the surface curve Γ_2, and $\mathbf{m}$ is defined by :

$$\mathbf{m}(\mathbf{p}_1(t)) = \mathbf{p}_2(t)\ . \tag{33}$$

When the space curve crosses the locus of centers of curvatures of the surface, the orthogonal projective tensor becomes singular. In our experiences, this singularity is detected by a change of sign of the determinant of the orthogonal projective tensor. When a singularity is detected, matrix conditioning is applied and the map is extended across the singular projection by tangent continuity [15].

In the example treated in Figures 6, 7 and 8 we shall demonstrate an implementation of orthogonal projection as it relates to the problem of curvature continuous blending. Given two primitive surface patches, the designer inputs a space curve, which is then projected onto both surfaces Figure 6. The orthogonal projection map is also used to define geometrically tangent directions on the surfaces along which the local surface curvatures are computed. Indeed the differential of the orthogonal projection (i.e. the orthogonal projective tensor) onto a surface, say $\mathbf{S}_1$, maps any vector at point $\mathbf{q}(t)$ into a vector in the tangent plane to the surface $\mathbf{S}_1$ at the footpoint $\mathbf{p}_1(t)$. In order to define a natural map, we chose to project onto $\mathbf{S}_1$ the vector $\overrightarrow{\mathbf{q}\mathbf{p}_2}$ joining $\mathbf{q}(t)$ to its projection $\mathbf{p}_2(t)$ onto the surface $\mathbf{S}_2$. The choice of this construction is justified by computational considerations and because it also applies at the boundary. The normal curvature of the surface $\mathbf{S}_1$ at the footpoint $\mathbf{p}_1(t)$ in the direction of the projection of $\overrightarrow{\mathbf{q}\mathbf{p}_2}$ are then used to

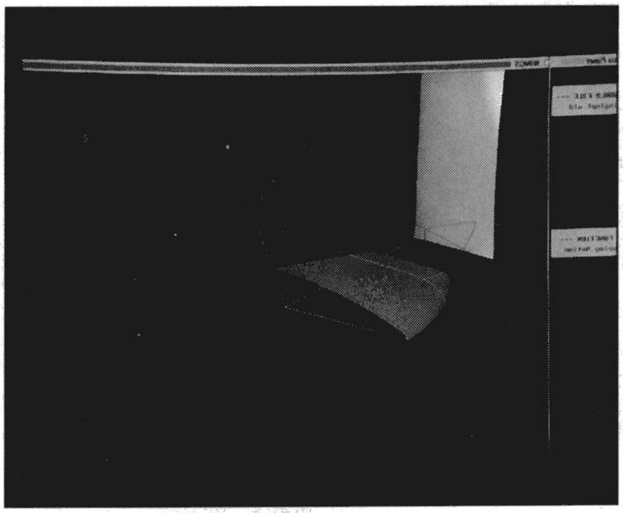

Fig. 6 Given two primitive surfaces the designer inputs a space curve, which is used as a design parameter to trim the surface patches, define a map between the two trimming curves and define a parameterization of the blending surface.

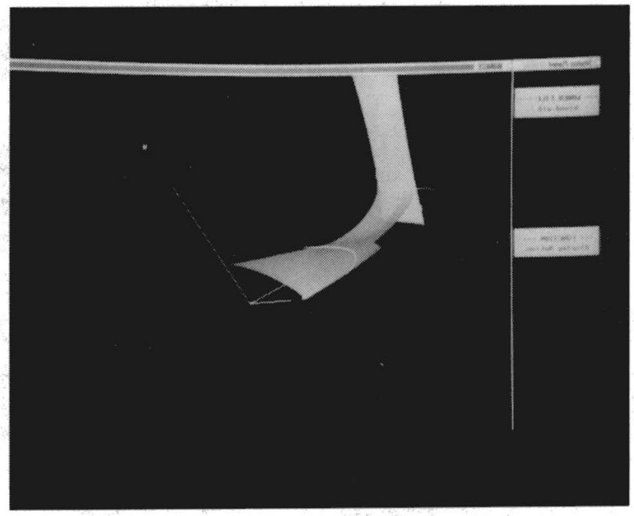

Fig. 7 The blend surface prior to trimming

define a sweeping curve whose envelope generates a curvature continuous blending surface (Figure 7.) Curvature continuity across the blend is guaranteed by the *Linkage Curve Theorem* established in [16]. In essence this theorem states a necessary and sufficient condition for curvature continuity across two surface patches that are normal continuous and connect along a common boundary curve called the *Linkage Curve. For two such surface patches to be linked curvature continuously it is sufficient that their normal curvatures agree at all points of the linkage curve along at least one direction, as long as this direction is not tangent to the linkage curve.* Finally, trimming is applied on the primitive surfaces yielding a new boundary that is common to the blend and the trimmed primitive surface patches Figure 8. For a more detailed description of the blending work and the treatment of orthogonal projection at the boundary the reader is referred to [17] and [18].

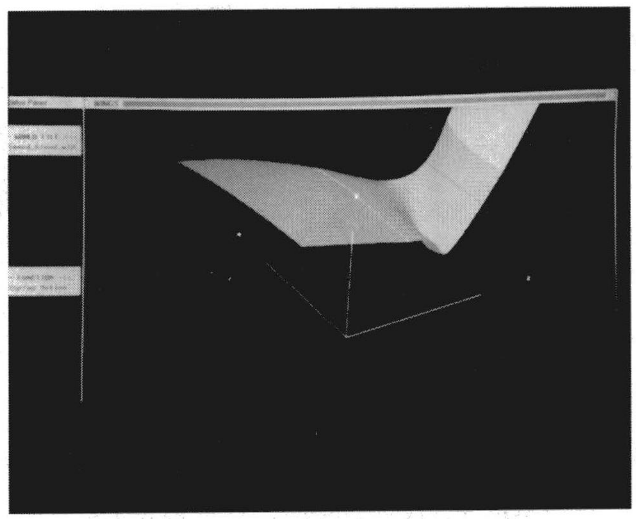

Fig. 8 The resulting surface patch after trimming.

7 Conclusion

This paper reports a new approach to the problem of designing curves on surfaces. The design solution proposed here uses an original geometric construction, *the orthogonal projection of a curve onto a free form surface*, which is used to map a space curve into a surface curve. This geometric construction can be formulated in terms of a differential equation, which is solved numerically. The orthogonal projection mapping was tested on a wide variety of surfaces defined implicitly or parametrically. Its practicality was demonstrated in the context of trimmed patches and curvature continuous surface blending.

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Figures 6, 7 and 8 were generated on a Silicon Graphics™ workstation by R.N. Raj and J. Pegna using the visualization software WINGS(©) written by A. Safi and J. Pegna. For the class of B-Spline curves and surfaces described in Section 4, the computation of the orthogonal projection curve (including its approximation employing [19]) has been implemented in the geometric modeling system Praxiteles developed at the MIT Design Laboratory.

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