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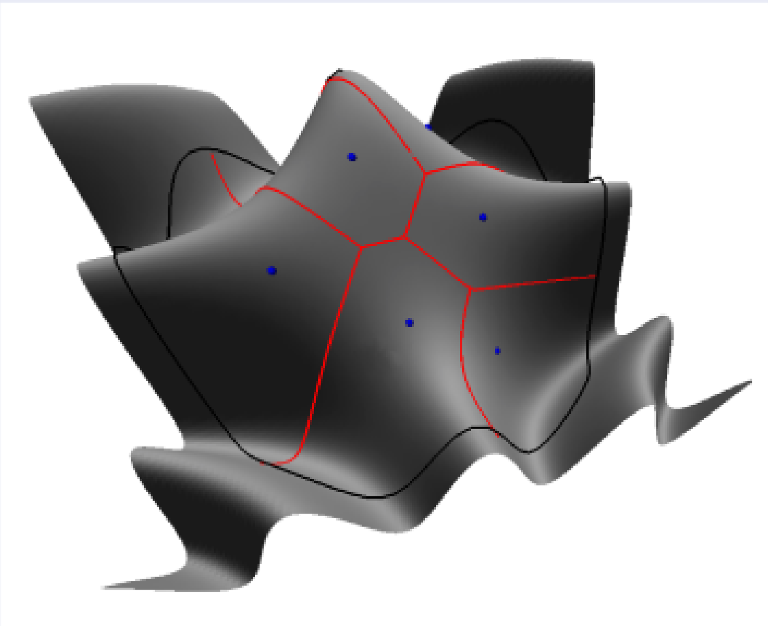
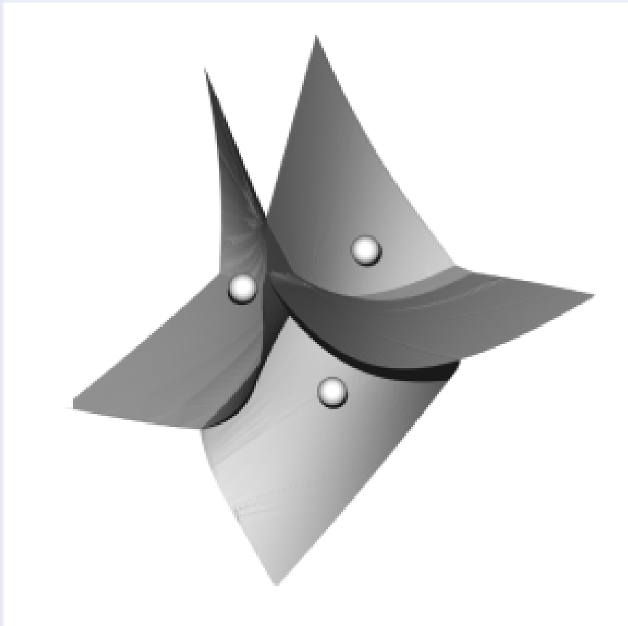
Computational Geometry in Euclidean and Riemannian Spaces

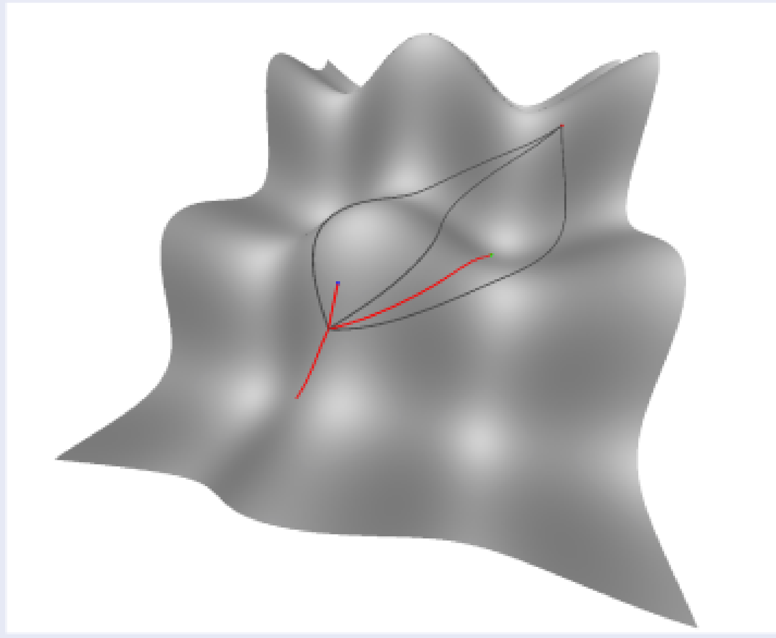
Distance computations and related concepts.



Riemannian VD in 2d



Riemannian VD in 3d



Cut Loci and Geodesics

General Case

For a connected and closed domain $G \subseteq M$ with boundary ∂G contained in a metric space (M, d) we define the *Medial Axis* (Fig. 1) as the closure of

$$\{m \in G : \text{ m has two shortest paths to } \partial G\}.$$

Alternatively the medial axis can be characterized via the closure of the centres m of maximal circles (spheres) with radius r that fit into G. Using the so called radius function

$$R : MA(G) \to R_{\geq 0} \quad m \mapsto r,$$

we can write the inverse medial axis transform (IMAT) as:

$$G = \bigcup_{m \in MA(G)} B\left(m, R(m)\right).$$

The medial axis is related to the concepts *Cut Locus* (Fig. 2) and *Symmetry Set* (Fig. 3) which define supersets of the medial axis or the well known *Voronoi Diagram* (Fig. 4):

- The Cut Locus $CL(D)$ of $D \subset M$ is the set of points in M, that have at least two shortest paths to D.
  $$MA(G) = CL(\partial G) \cap G$$
- The Symmetry Set $SY(D)$ is a local counterpart of the Cut Locus requirung two locally shortest paths to D.
  $$MA(D) \subseteq CL(\partial D) \subseteq SY(\partial D)$$
- The Cut Locus for a set of Points $P = \{p_1, \dots, p_n\}$ is known as the Voronoi Diagram $VD(P)$ of P.
  $$VD(P) = CL(P)$$

A special case of a metric space is the euclidean space $\mathbb{R}^n$ together with the metric $d(x, y) := \|x - y\|$ for $x, y \in \mathbb{R}^n$. The following gives some two-dimensional examples:

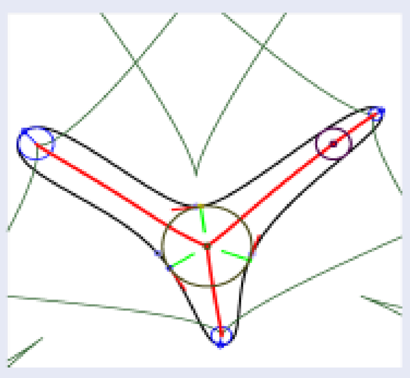


Fig. 1

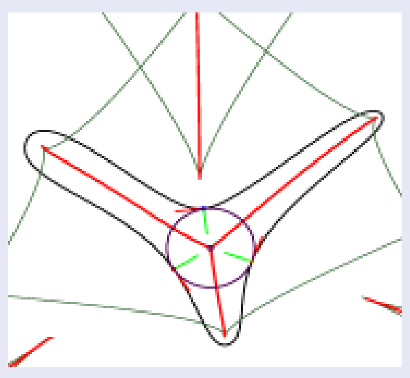


Fig. 2

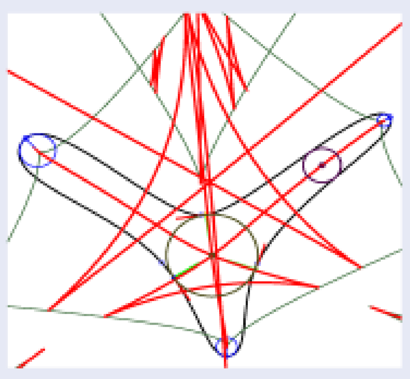


Fig. 3

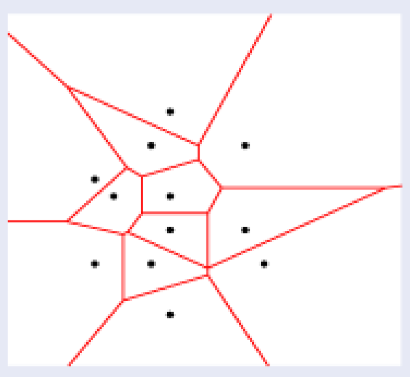


Fig. 4

Riemannian Case

The definitions above use metric spaces since all concepts make use of the distance function d. A metric space with geometric background is given by a *Riemannian Manifold* which is a differentiable manifold together with a so called *metric tensor* (g_{ij}). The metric tensor provides an additional structure that enables us to measure local distances and angles on the manifold. As a special case we consider $M \subset \mathbb{R}^3$ defined via

$$M = \{(u, v, h(u, v)) : (u, v) \in U\},$$

using a height function $h : U \subset \mathbb{R}^2 \to \mathbb{R}$. In this case the metric tensor can be explicitly written as

$$g_{11} = 1 + \left(\frac{\partial h}{\partial u}\right)^2 \quad g_{12} = \frac{\partial h}{\partial u} \cdot \frac{\partial h}{\partial v} \quad g_{22} = 1 + \left(\frac{\partial h}{\partial v}\right)^2.$$

The metric tensor induces the so called *Riemannian metric*:

$$d(p, q) = \inf\{L(c) : c \text{ curve in } M \text{ that connects } p \text{ with } q\}$$

Here we make use of the length $L(c)$ of a surface curve c. A curve that realizes the distance $d(p, q)$ is called a shortest path. Unfortunately these global shortest paths are usually inaccessible to analytic methods from the field of differential geometry. However, locally shortest paths (so called *Geodesics*) can be computed using the *Geodesic ODE-system*:

$$u_k'' + \sum_{i,j=1}^{2} \Gamma_{ij}^k u_i' u_j' = 0 \quad (k = 1, 2).$$

The *Christoffel symbols* Γ_{ij}^k can be directly obtained from the Riemannian metric tensor (g_{ij}). Much effort has been put into research on computing the aforementioned concepts on two- or three-dimensional height surfaces:

- Fig. 5: Medial Axis Transform on a 2d height surface.
- Fig. 6: Inverse Medial Axis Transform on a 3d hyper surface embedded in $\mathbb{R}^4$, drawn in the parametrer space U.

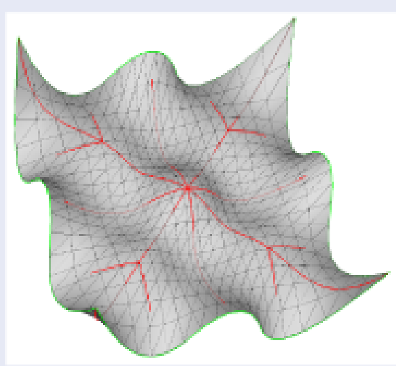


Fig. 5

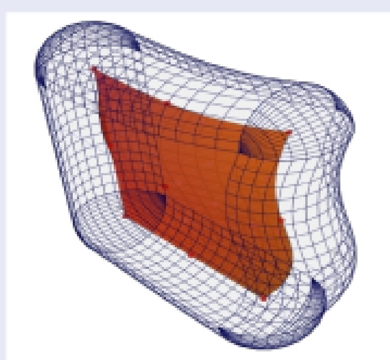


Fig. 6

Distance computations

The problem of computing the distance $d(p, q)$ of two arbitrary points $p, q \in M$ can be reduced to finding all geodesics that connect p and q. To adress this problem we let $O_p(s, \varphi)$ denote the endpoint of the unique geodesic starting at p in the direction parametrized by an angle parameter φ with length s and call $O_p : \mathbb{R} \times [0, 2\pi) \to M$ *geodesic offset function.*
Using this notation the shortest path problem is expressed as the problem of finding all roots (s, φ) of the function

$$F(s, \varphi) := O_p(s, \varphi) - q.$$

A geometrically motivated homotopy approach is given by

$$H(s, \varphi, \lambda) := O_p(s, \varphi) - q(\lambda),$$

where $q : [0, 1] \to M$ is a C^1-curve with $q(\xi) = q$. This curve can be chosen arbitrarily to a large extent, taking into account the focal curves with respect to p. Starting from the point $(s_0, \varphi_0, 0)$ with known s_0, φ_0 given by $O_p(s_0, \varphi_0) = q(0)$, we trace the zero curve $H^{-1}(0)$ collecting all points (s, φ, λ) with $\lambda = \xi$.

- Fig. 7: Three different geodesics that connect p and q.
- Fig. 8: Projection of $H^{-1}(0)$ into the φ-λ-plane.

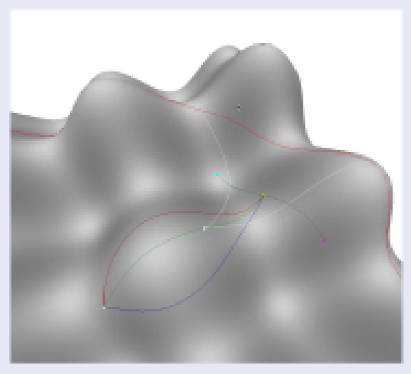


Fig. 7

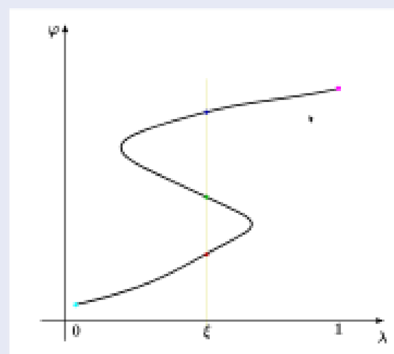


Fig. 8

In this case we have computed all geodesics that connect p and q, i.e. we have parameters $(s_1, \varphi_1), (s_2, \varphi_2), (s_3, \varphi_3)$ with $O_p(s_i, \varphi_i) = q$ $(i = 1, 2, 3)$. Since the shortest path is always a geodesic we obtain the shortest path $O_p(\cdot, \varphi) : [0, s] \to M$ by choosing $s = min_i\{s_i\}$ and φ accordingly. By the definition of our metric we have

$$d(p, q) = s.$$

References

- Naß, H.; Wolter, F.-E.; Dogan, C.; Thielhelm, H., "Medial Axis Inverse Transform in 3-Dimensional Riemannian Complete Manifolds", in Proceedings of NASAGEM (2007), 386–395
- Naß, H.; Wolter, F.-E.; Thielhelm, H.; Dogan, C., "Computation of Geodesic Voronoi Diagrams in 3-Space using Medial Equations", in Proceedings of NASAGEM (2007), 376–385

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